ABSTRACT
The present paper develops an analytical solution for piezoelectric laminated composite plates subjected to electro-mechanical loading. The displacements and stresses are the important parameters which will influence for usage of laminated plates. The object of the present study is to present analytical formulations and solutions to study the bending behavior of cross ply laminated composite plates embedded with piezoelectric fiber reinforced composite (PFRC) material using higher order theory. This study considers the thickness stretching effect and non-zero transverse shear stresses conditions on the top and bottom surfaces of the plate. The equations of equilibrium and boundary conditions are derived using the principle of virtual work. Solutions are obtained for smart composite plate in closed-form using Navier's technique. Comparison studies are performed to verify the validity of the present results. The results predicted by present theory are close agreement with the exact solutions. The effect of aspect ratio and voltage on the in-plane and transverse displacements and also on the normal and transverse shear stresses has been investigated.

KEYWORDS: Smart structures, Higher Order Shear Deformation Theory, Piezoelectric fiber composites, Naviers solutions.

INTRODUCTION
Laminated composite structures, with embedded piezoelectric actuators and sensors combines some of the superior mechanical properties of composites with the additional capabilities to sense deformations and stress states and to adapt their response accordingly. Piezo-laminates as smart-intelligent composites offer great potential for active control of advanced aerospace, nuclear, and automotive structural applications. In the past, Jafar Rouzegar and Abad [1], presented analysis of cross ply laminates with PFRC actuators using four variable refined plate theory. Raju, and Suresh Kumar [2], investigated the buckling analysis of smart material plates using higher order theory, they developed an analytical procedure to investigate the buckling characteristics of smart material plates subjected to electromechanical loading based on higher order theory. Shiyekar and Kant [3], presented a complete analytical solution for cross ply composites integrated with piezoelectric fiber reinforced composite (PFRC) actuators under bi-directional bending. They used Higher Order Shear Deformation Normal Theory with 12 degrees of freedom to analyze the hybrid laminates subjected to electromechanical loading. Ray et al [4], presented an exact analysis of piezoelectric plate under cylindrical bending. They derived the solutions for a simply supported plate subjected to sinusoidal mechanical loading and electrical potential on the top surface and zero electric potential at the edges. They recommended a linear variation of deformations and electric potential across the thickness. Mallik and Ray [5] presented exact
solutions for analysis of piezoelectric fiber reinforced composites as distributed actuators for smart composite laminate plates. Reddy [6], developed the theoretical formulations, the Navier solutions and finite element models based on classical and shear deformation plate theories to study the laminated composite plate integrated with sensors and actuators under mechanical and electrical loadings. He used the negative velocity feedback control algorithm coupling the direct and converse piezoelectric effects to actively control the dynamic response of an integrated structure through closed loop control. Heyliger [7], investigated the static behavior of laminated elastic / piezoelectric plates. Mitchel and Reddy [8], presented a higher order shear deformation theory for composite laminates with piezoelectric lamina. Manjunath and, Kant [9], given a New theories for symmetric/unsymmetrical composite and sandwich beams with C0 finite elements. Kant and Swaminathan [10], presented Analytical solutions for the static analysis of laminated composite and sandwich plates based on a higher order refined theory. Pagano [11] presented exact solutions for composite laminates in cylindrical bending. Crawley and De Luis [12] has given the use of piezoelectric actuators as elements of intelligent structures. Robbins and Reddy [13] studied the analysis of piezoelectrically actuated beams using a layer wise displacement theory. Vel and Bra[14] presented, Cylindrical bending of laminated plates with distributed and segmented piezoelectric actuators/ sensors. Batra and Liang [15], investigated a Finite dynamic deformations of smart structures. Lee [16] developed a Theory for laminated piezoelectric plates for the design of distributed sensors/actuators.

In the present work, analytical formulations and solutions are presented to study the bending behavior of cross ply laminated composite plates embedded with piezoelectric fiber reinforced composite (PFRC) material using higher order theory with 11 unknown variables. This study considers the thickness stretching effect and non-zero transverse shear stresses conditions on the top and bottom surfaces of the plate. The equations of equilibrium and boundary conditions are derived using the principle of virtual work. Solutions are obtained for a smart composite plate in closed-form using Navier’s technique. Comparison studies are performed to verify the validity of the present results. The results predicted by present theory are close agreement with the exact solutions. The effect of aspect ratio and voltage on the in-plane and transverse displacements and also on the normal and transverse shear stresses has been investigated.

Theoretical Formulation:

A simply supported rectangular cross ply laminated substrate plate is considered for flexural analysis as shown in Fig.1.A layer of piezoelectric fiber reinforced composite (PFRC) material is embedded to the top surface of the plate, which is acting as the distributed actuator of the plate. The ‘z’ axis is assumed at the middle of the plate i.e. it is located at the distances +h/2 and –h/2 from the top and bottom of the composite cross ply laminate. The thickness of the PFRC actuator is assumed as t0. The displacement vectors u(x,y,z), v(x,y,z) and w(x,y,z) at any point in the laminate are expanded in the powers of ‘z’ axis. The displacement vectors are expanded in the following form.

\[ u(x,y,z) = u_0(x,y) + z\theta_y(x,y) + z^2 u_2(x,y) + \ldots \]
\[ v(x,y,z) = v_0(x,y) + z\theta_x(x,y) + z^2 v_2(x,y) + \ldots \]
\[ w(x,y,z) = w_0(x,y) + z\theta_z(x,y) + z^2 w_2(x,y) + \ldots \]

Fig. 1: Geometry of laminated composite plate with simply supported all edges with PFRC actuator at top.

Where the parameters \(u_0, v_0\) denotes the in-plane displacements and \(w_0\) is the transverse displacement at any point on the mid plane of the substrate. The functions \(\theta_x, \theta_y\) are the rotations of the normal to the mid plane about y and x axes respectively. The remain parameters \(u_0,v_0,w_0,\theta_x,\theta_y\), and \(\theta_z\) are the respective higher order parameters related to the transverse deformation modes.
2.1. Lamina coupled constitutive equations:

The linear constitutive relations for a single piezoelectric layer couples the elastic and electric fields as given below,

\[
\{\sigma\} = [Q] \{\varepsilon\} - [e] \{E\},
\]

\[
\{D\} = [e^*] \{\varepsilon\} - [\eta] \{E\}.
\]

The elastic field intensity vector \(E\) related to electrostatic potential \(\xi(x,y,z)\) in the \(i^{th}\) layer is given by

\[
E_x^i = \frac{\delta(x,y,z)}{\delta x}; \quad E_y^i = \frac{\delta(x,y,z)}{\delta y}; \quad E_z^i = \frac{\delta(x,y,z)}{\delta z}
\]

Where \(\sigma, Q, \varepsilon, e, E, D\) and \(\eta\) are the stress vector, elastic constant matrix, strain vector, piezoelectric constant matrix, electric field intensity vector, electric displacement vector and dielectric constant matrix respectively. From the lamina coupled constitutive equations, the elastic field can be written as two components of stresses. The first is elastic stress component \((es)\) and second is piezoelectric stress component \((pz)\).

i.e \(\{\sigma\} = \{\sigma\}^{es} - \{\sigma\}^{pz}\),

where

\[
\{\sigma\}^{es} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} \\
Q_{12} & Q_{22} & Q_{23} & Q_{24} \\
Q_{13} & Q_{23} & Q_{33} & Q_{34} \\
Q_{14} & Q_{24} & Q_{34} & Q_{44}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{x} \\
\varepsilon_{y} \\
\varepsilon_{z}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} (6)
\]

\[
\{\sigma\}^{pz} = \begin{bmatrix}
e_{31} \\
e_{32} \\
e_{33} \\
e_{15}
\end{bmatrix}
\]

In which \(\{\sigma\}, \{\varepsilon\}\) are the stresses and linear strain vectors with respect to the laminate axes \(x, y, z\), and \(Q_{ij}\)s are the plane stress reduced elastic constants in the plate laminate axes of the \(i^{th}\) lamina. The resultant stress equations are defined as elastic and piezoelectric stress resultants as

Elastic stress resultants: \([Q S M N]^{es}\):

\[
[Q_{x}^{es}, Q_{y}^{es}, Q_{z}^{es}] = \sum_{l=1}^{n} Z_{l}^{es} [l_{x}^{es}, l_{y}^{es}, l_{z}^{es}] \int \frac{z}{z^2} dz \quad S_{x}^{es}, S_{y}^{es}, S_{z}^{es} = \sum_{l=1}^{n} Z_{l}^{es} [l_{x}^{es}, l_{y}^{es}, l_{z}^{es}] \int \frac{z}{z^2} dz
\]

Piezoelectric stress resultants: \([Q S M N]^{pz}\):

\[
[Q_{x}^{pc}, Q_{y}^{pc}, Q_{z}^{pc}] = \int \frac{z}{z^2} dz \quad S_{x}^{pc}, S_{y}^{pc}, S_{z}^{pc} = \int \frac{z}{z^2} dz
\]

Total stress resultants: \([Q S M N] = [Q S M N]^{es} + [Q S M N]^{pz}\)
1.2. Admissible boundary conditions for the Navier’s solutions of the displacement model:

To discuss the Navier and other solutions, consider the governing equations of motion in terms of the displacements for the higher order displacement model. In Navier method the displacements are expanded in terms of unknown parameters. The choice of the trigonometric functions in the series is restricted to those which satisfy the boundary conditions of the problem. The Navier solutions can be developed for the rectangular laminates with two sets of simply supported boundary conditions. The following are the mechanical and electrical in plane boundary conditions for the simply supported plate.

At edges \( x=0 \) and \( x=a; \) \( v_{0*}=0, M_{x0*}=0, N_{x0*}=0, \theta_{x0*}=0, M_{y0*}=0, N_{y0*}=0, \xi_{0*}=0, \)\( \eta_{0*}=0, \) \( y=0 \) and \( y=b; \) \( v_{0*}=0, M_{y0*}=0, N_{y0*}=0, \theta_{y0*}=0, M_{x0*}=0, N_{x0*}=0, \xi_{0*}=0, \)\( \eta_{0*}=0, \)

By considering the above boundary conditions and using the Navier’s method, the mechanical load, electrical load and mid plane displacements are expanded as follows:

\[
\begin{align*}
U_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{0mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi x}{b}, \\
V_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{0mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{b}, \\
W_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{0mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi x}{b}, \\
\theta_x &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{xmn} \cos \frac{m\pi x}{a} \sin \frac{n\pi x}{b}, \\
\theta_y &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{ymn} \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{b}, \\
q_z &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{zmn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b},
\end{align*}
\]

And electrostatic potential is given as

\[
\zeta(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \zeta_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b},
\]

(9)

The above expansions are substituted in to the governing equation of motions and which gives the required system equations as per the Kant and Swaminathan (2002).

RESULTS AND DISCUSSION

In this analysis a hybrid cross ply rectangular laminated composite plate with all sides are simply supported is to be considered. The plate contains polymer based layers of graphite/epoxy composite with bidirectional orthotropic. At the top of the elastic substrate a Piezoelectric Fiber Reinforced Composite material (PFRC) is layered in a distributed form. The following material properties are taken for the elastic substrate;

\begin{align*}
E_1 &= 172.9 \text{ Gpa};& E_{12} = 25; & \quad G_{12} = 0.5 \text{ E}_{12}; & \quad G_{22} = 0.2 \text{ E}_{12}; & \quad v_{12} = 0.25; \\
\end{align*}

The subscripts ‘1’ denotes the longitudinal direction of fiber and ‘2’ denotes the transverse direction of fiber. The material properties of PFRC layer taken as

\begin{align*}
C_{11} &= 32.6 \text{ Gpa}; & \quad C_{12} = C_{21} = 4.3 \text{ Gpa}; & \quad C_{13} = C_{31} = 4.76 \text{ Gpa}; & \quad C_{22} = C_{32} = 7.2 \text{ Gpa}; & \quad C_{33} = 3.85 \text{ Gpa}; \\
C_{44} &= 1.05 \text{ Gpa}; & C_{55} = C_{66} = 1.29 \text{ Gpa}; & C_{15} = -6.76 \text{ C/m}^2; \\
\eta_{11} &= \eta_{22} = 0.037 \text{ E-9} \text{ C/Vm}; & \quad \eta_{33} = 10.64 \text{ E-9} \text{ C/Vm}; \\
\end{align*}

The mechanical and electrical loads are represented by the following general form of sinusoidal loads;

\[
q_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 40 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \zeta \left( \frac{h}{2} + t_p \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 100 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b},
\]
The various values of aspect ratios of plate(S=a/h), i.e S=10,S=20 and S=100 are considered for the testing and the thickness of the actuating layer is assumed as 250μm and the thickness of the each orthotropic layer is ‘1’mm. A three layered symmetric (0°/90°/0°) composite laminate is to be considered for the present work. The numerical results with non-dimensional manner are calculated and compared with the 3D exact solutions.

In Table1, the normalized in plane and transverse displacements are labeled for laminate (0°/90°/0°) for various aspect ratios(S=10,S=20 and S=100). The % of error is calculated as per the literature available and the obtained results are appreciable compared with the exact solution results. It is observed that the actuating voltage is more effective in the case of thick laminates compared with thin laminates. The present results are more excellent in all types of loading cases and various aspect ratios. The better in the obtained results is due to the assumption of cubical variation in the approximation of in plane as well as transverse displacements (u, w). The variation of displacements with respect to various aspect ratios is showed in Fig.1 and 2. It is observed that for thin plate (S=100), the variations for displacements are linear and constant with the thickness of the laminate for both in plane and transverse displacements.

The numerical results for the in plane and transverse normal stresses, and shear stresses are showed in non-dimensional manner in Table2 and 3 respectively with different loadings(V=0,V=+100, V=100) and different aspect ratios. It is observed that the in plane normal stress (σx) is more than the in plane normal stress(σy). The variations of in plane normal stress with different aspect ratios at V=100 are showed in Fig 3. For the moderately thin plates (S=20) the % of variations in the in plane and transverse displacements are 1.2% and 0.73% respectively with applied electrical loading.

**Conclusions:**

The present paper gives the complete solution for the laminated composite plates embedded with piezoelectric actuator, subjected to different electromechanical loadings. The higher order shear and normal deformation theory has been developed and solutions are obtained in closed form with Navier’s technique. The numerical results are obtained for displacements and stresses induced in the laminate with the applied electrical and mechanical loading. It is observed that the actuating effects are more in the case of thick lamitnates rather than the thin laminates. By the comparison the present results are effective and more reliable with the exact solutions results available in literature. The scope of future work is by using the kinematical assumptions provided useful for developing analytical solutions for vibration and transient analysis of cross ply piezoelectric laminated plates.

### Table 1: Normalized in plane and transverse displacements of \( \vec{U}, \vec{W} \) of symmetric substrate (0°/90°/0°) without and with sinusoidal electric voltages at top of the PFRC actuator surface.

<table>
<thead>
<tr>
<th>Theory</th>
<th>S=10</th>
<th>S=20</th>
<th>S=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>V=0</td>
<td>V=+100</td>
<td>V=100</td>
</tr>
<tr>
<td>( \vec{U} = (0, \frac{b}{2}, \pm \frac{h}{2}) )</td>
<td>0.0040</td>
<td>-3.1623</td>
<td>3.1704</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0066</td>
<td>-3.1410</td>
<td>3.1542</td>
</tr>
<tr>
<td>( \vec{W} = (\frac{a}{2}, \frac{b}{2}, 0) )</td>
<td>-0.6817</td>
<td>130.55</td>
<td>-131.9</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.7100</td>
<td>132.90</td>
<td>-134.3</td>
</tr>
</tbody>
</table>

### Table 2: Normalized in plane and transverse normal stresses of \( \sigma_x, \sigma_y, \sigma_z \) of symmetric substrate (0°/90°/0°) without and with sinusoidal electric voltages at top of the PFRC actuator surface.

<table>
<thead>
<tr>
<th>Theory</th>
<th>S=10</th>
<th>S=20</th>
<th>S=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>V=0</td>
<td>V=+100</td>
<td>V=100</td>
</tr>
<tr>
<td>( \sigma_x = (\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2}) )</td>
<td>-0.517</td>
<td>247.85</td>
<td>-248.88</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.528</td>
<td>248.76</td>
<td>-249.82</td>
</tr>
<tr>
<td>( \sigma_y = (\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2}) )</td>
<td>-0.249</td>
<td>40.659</td>
<td>-41.157</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.257</td>
<td>42.532</td>
<td>-43.046</td>
</tr>
<tr>
<td>( \sigma_z = (\frac{a}{2}, \frac{b}{2}, 0) )</td>
<td>-0.534</td>
<td>23.838</td>
<td>-44.907</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.480</td>
<td>-0.485</td>
<td>-0.5114</td>
</tr>
</tbody>
</table>
Table 3: Normalized in plane and transverse shear stresses of \( \tau_{xy}, \tau_{yz}, \tau_{xz} \) of symmetric substrate (0/90/0°) without and with sinusoidal electric voltages at top of the PFRC actuator surface.

<table>
<thead>
<tr>
<th>Theory</th>
<th>S=10</th>
<th>S=20</th>
<th>S=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_{xy} = \left( 0,0,\pm \frac{a}{2} \right) ) Present</td>
<td>V=0</td>
<td>V=+100</td>
<td>V=-100</td>
</tr>
<tr>
<td>Present</td>
<td>0.0256</td>
<td>-7.652</td>
<td>7.704</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0261</td>
<td>-7.696</td>
<td>7.748</td>
</tr>
<tr>
<td>( \tau_{yz} = \left( 0,\frac{a}{2},0 \right) ) Present</td>
<td>-0.1101</td>
<td>22.035</td>
<td>-22.23</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.121</td>
<td>24.065</td>
<td>-24.307</td>
</tr>
<tr>
<td>( \tau_{xz} = \left( \frac{a}{2},0,0 \right) ) Present</td>
<td>-0.246</td>
<td>19.896</td>
<td>-20.389</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.344</td>
<td>25.609</td>
<td>-24.298</td>
</tr>
</tbody>
</table>

Fig. 2: Variation of Nondimensional inplane displacements against aspect ratio

Fig. 3: Variation of Nondimensional transverse displacements against aspect ratio
Fig. 4: Variation of Nondimensional normal stresses against aspect ratio at voltage, V=100

REFERENCES

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