Flow Behavior of Boundary Layer Flow of Nanofluid over a Flat Plate Using HAM

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ABSTRACT
An analysis has been performed to study the problem of boundary layer flow of nanofluids over a flat plate. The governing equations for this problem are reduced to an ordinary form and is solved using Homotopy Analysis Method (HAM) and numerically by fourth order Runge–Kutta technique. Also, Velocity fields have been computed and shown graphically for various values of physical parameters. The main objective of the present work is to investigate the effect of the nanoparticle volume fraction on the velocity field.

INTRODUCTION
The classical concept of boundary layer corresponds to a thin region next to the wall in a flow where viscous forces are important which may affect the engineering process of producing. For example, viscous forces play essential roles in glass fiber drawing, crystal growing and plastic extrusion. Blasius [1] studied the simplest boundary layer over a flat plate. He employed a similarity transformation which reduces the partial differential boundary layer equations to a nonlinear third-order ordinary differential one before solving it analytically. In contrast to the Blasius problem, Sakiadis [2] introduced the boundary layer flow induced by a moving plate in a quiescent ambient fluid. A large amount of literatures of this problem has been cited in the books by Schlichting and Gersten[3], Bejan[4]and White[5]and also in research papers such as[6,7].

As technology improves, it was realized that the industrial devices have to be cooled in more effective ways [8] and the conventional fluids such as water are not appropriate anymore, so the idea of adding particles to a fluid was presented. Adding nano-particles to a base fluid affects the homogeneity of the fluid and the randomness motion of the molecules increases. These tiny particles have high thermal conductivity, so the mixed fluids have better thermal properties [9–11]. The materials of these nano-scale particles are aluminum oxide (Al2O3), copper (Cu), copper-oxide (CuO), gold (Au), silver (Ag), etc., which are suspended in base fluids such as water, oil, acetone and ethylene glycol. Al2O3 and CuO are the most well-known nano-particles used by many researchers in their studies [11–14].

Most scientific problems in fluid mechanics and heat transfer problems are inherently nonlinear. In recent decades many attempts have been made to develop analytical methods for solving such nonlinear equations. One of them is the perturbation method [15], which is strongly dependent on a so called small parameter to be defined according to the physics of the problem. Thus, it is worth developing some new analytical techniques, which are independent of defining a small parameter such as Homotopy Perturbation Method (HPM) [16, 17], Variational Iteration Method (VIM) [18]. In fact the perturbation method cannot provide a simple way to adjust and control the region and rate of convergence of a particular approximated series. Liao introduced the basic idea of Homotopy in topology to propose a general analytical method for nonlinear problems, namely the Homotopy Analysis Method [19, 20], that does not need any small parameter. This method has been successfully applied to solve many types of nonlinear problems [21-24].
The aim of this study is to investigate the effect of physical parameters on boundary layer flow of nano-fluids over a flat plate. The novelty of this study is considering the effects of nano-particle inclusion in the classical Blasius problem. The convergence of the series solution is also explicitly discussed. Obtaining the analytical solution of the models and comparing with numerical result reveal the capability, effectiveness and convenience of HAM.

1. Problem statement and mathematical formulation:

We consider an incompressible viscous nanofluid which flows over a flat plate, as shown in Fig. 1. It is assumed that the free stream velocity, \( U_\infty \), is uniform and constant. Further, the flow in the laminar boundary layer is two-dimensional [14].

Fig. 1: Velocity boundary layers for a nanofluid over a flat plate.

It is also assumed the thermal properties of above-mentioned nanofluid are temperature-independent and the fluid and nanoparticles are in thermal equilibrium and no slip occurs between the particles and water; as a result; nanofluid acts as a conventional homogeneous single-phase fluid. Regarding these assumptions, the continuity, momentum and energy equations can be expressed as:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{2}
\]

And the appropriate boundary conditions are:

\[
u |_{y=0} = 0, \quad \nu |_{y=0} = 0, \quad \lim_{y \to \infty} u = U_\infty \tag{3}
\]

Let us introduce some physical properties of the nanofluids [25]

\[
\mu_{nf} = \frac{\mu_f}{(1-\phi)^2}, \quad \rho_{nf} = (1-\phi)\rho_f + \rho_p \tag{4}
\]

Where \( \mu_{nf} \) is the viscosity of the nanofluid, \( \phi \) is the solid volume fraction of the nanofluid, \( \rho_f \) is the density of the base fluid, \( \rho_p \) is the density of the solid particle and \( \mu_f \) is the viscosity of the base fluid and thermal properties are:

\[
\left( \rho c_p \right)_{nf} = (1-\phi)\left( \rho c_p \right)_f + \phi \left( \rho c_p \right)_p \tag{5}
\]

\[
\frac{k_{nf}}{k_f} = \frac{(k_f + 2k_p) - 2\phi(k_f - k_p)}{(k_f + 2k_p) + \phi(k_f - k_p)} \tag{6}
\]

Here \( k_f \) is the thermal conductivity of the fluid, \( k_p \) is the thermal conductivity of the solid, and \( k_{nf} \) is thermal conductivity of the nanofluid. \( (\rho c_p)_{nf}, (\rho c_p)_f, (\rho c_p)_p \) are heat capacity of the nanofluid, fluid and particle respectively. Eqs.(4) and (5) are valid for spherical particles and its values for different materials are listed in Table 1.

<table>
<thead>
<tr>
<th>Material</th>
<th>Density ((kg/m^3))</th>
<th>Heat capacity ((J/kg K))</th>
<th>Thermal Conductivity ((W/m K))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Al_2O_3 )</td>
<td>3970</td>
<td>765</td>
<td>40</td>
</tr>
</tbody>
</table>
can be expanded in a power series of
\(\phi\) 
\(\frac{\partial}{\partial x} f(\eta)\) 
\(\frac{\partial}{\partial y} f(\eta)\) 

Where \(\psi\) is the usual stream function, i.e. \(u = \frac{\partial \psi}{\partial y}\) and \(v = -\frac{\partial \psi}{\partial x}\) is the kinematic viscosity of the base fluid. Substituting Eqs.(7) into Eq. (2) we obtained the following ordinary differential equation:

\[
1 - (1 - \phi)^{2}(1 - \phi + \phi \frac{\rho_{w}}{\rho}) \eta \frac{d^2 f}{d\eta^2} + (1 - \phi - \phi^2 + \phi \frac{\rho_{w}}{\rho}) \eta \frac{d f}{d\eta} = (1 - \phi)^{2} (1 - \phi + \phi \frac{\rho_{w}}{\rho}) \epsilon \frac{\partial^2}{\partial x^2} f(\eta) + (1 - \phi)^{2} (1 - \phi + \phi \frac{\rho_{w}}{\rho}) \epsilon \frac{\partial^2}{\partial y^2} f(\eta) = 0
\]

With these boundary conditions

\[
f_{f}(0) = 0, \quad \frac{df}{d\eta} \big|_{\eta=0} = 0, \quad \lim_{\eta \to \infty} \frac{df}{d\eta} = 1
\]

2. Implementation of the Homotopy Analysis Method:

For HAM solutions, we choose the initial guess and auxiliary linear operator in the following form:

\[
f_{\phi}(\eta) = \frac{1}{(1 - \phi + \phi \frac{\rho_{w}}{\rho}) \eta} \left[ (1 - \phi - \phi^2 + \phi \frac{\rho_{w}}{\rho}) \epsilon \frac{d^2 f}{d\eta^2} + (1 - \phi)^{2} (1 - \phi + \phi \frac{\rho_{w}}{\rho}) \epsilon \frac{d f}{d\eta} \right]
\]

\[
L(f) = \frac{1}{(1 - \phi + \phi \frac{\rho_{w}}{\rho}) \eta} \frac{d^2 f}{d\eta^2} (\eta) + \frac{d^2 f}{d\eta^2} (\eta)
\]

\[
L(C_{1} + C_{2} \eta + C_{3} \epsilon) = 0
\]

Where \(C_{i}(i = 1, 2, 3)\) are constants. Let \(p \in [0, 1]\) denotes the embedding parameter and \(\hat{t}\) indicates non-zero auxiliary parameters. We then construct the following equations:

Zeroth -order deformation equations

\[
(1 - p)L[F(y; p) - f_{0}(y)] = ph(y)N[F(y; p)]
\]

\[
F(0; p) = 0; \quad F'(0; p) = 0, \quad F''(0; p) = 1
\]

\[
N[F(\eta; p)] = \frac{1}{(1 - \phi + \phi \frac{\rho_{w}}{\rho}) \eta} \frac{d^2 f}{d\eta^2}(\eta) + \frac{d^2 f}{d\eta^2}(\eta)
\]

For \(p = 0\) and \(p = 1\) we have

\[
F(\eta; 0) = f_{0}(\eta) \quad F(\eta; 1) = f(\eta)
\]

When \(p\) increases from 0 to 1 then \(F(\eta; p)\) varies from \(f_{0}(\eta)\) to \(f(\eta)\). By Taylor’s theorem and using equation (16), \(F(\eta; p)\) can be expanded in a power series of \(p\) as follows:

\[
F(\eta; p) = f_{0}(\eta) + \sum_{n=1}^{\infty} f_{n}(\eta) p^n \quad f_{n}(\eta) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \bigg|_{p=0} F(\eta; p)
\]
In which $\hat{h}$ is chosen in such a way that this series is convergent at $p = 1$, therefore we have through equation (17) that

$$f(\eta) = f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta).$$

$m$th-order deformation equations

$$L\left[f_m(\eta) - \chi_m f_{m+1}(\eta)\right] = h H(\eta) R_m(\eta)$$

$$F_m(0; p) = 0; \quad F_m(0; p) = 0, \quad F_m(\infty; p) = 0$$

$$R_m(\eta) = \frac{1}{((1-\phi) + \frac{\rho_f}{\rho})^2} f_m^0 + \sum_{n=0}^{\infty} \left( f_{m+1} + f_n \right)$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now we determine the convergency of the result, the differential equation, and the auxiliary function according to the solution expression. So let us assume:

$$H(\eta) = e^{-\eta}$$

We have found the answer by maple analytic solution device. The first deformation is presented below when $\rho_f = 3970, \rho_s = 9971$ and $\phi = 0$

$$f_1(\eta) = e^{-2\eta} - e^{-\eta} \left( 2000000000 e^{-21} + 390405 e^{-14} \right)$$

The solutions $f(\eta)$ were too long to be mentioned here, therefore, they are shown graphically.

3. Convergence of the HAM solution:

As pointed out by Liao [18], the convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter $h$. To influence of $h$ on the convergence of solution, we plot the so-called $h$-curve of $f''(0)$, as shown in Figures. 2-4.

![Fig. 2: The $h$ - validity for particle Cu and different value of volume fraction $\Phi$.](image)

![Fig. 3: The $h$ - validity for particle TiO$_2$ and different value of volume fraction $\Phi$.](image)

The solutions converge for $h$ values which are corresponding to the horizontal line segment in $h$ curve. In order to investigate the range of admissible values of the auxiliary parameter $h$, for various quantities...
of $\text{Cu}_6\text{TiO}_3$ and $\text{Al}_2\text{O}_3$, the curves of $\hbar$ were derived 6th-order approximations. Figures 2-4 shows a typical $\hbar$ curve for $f'^{(0)}$ that shows admissible values for auxiliary parameter $\hbar$.

![Graph showing $\hbar$ as a function of $f'^{(0)}$](image)

**Fig. 4: The $\hbar$ - validity for particle $\text{Al}_2\text{O}_3$ and different value of volume fraction $\Phi$**

**RESULTS AND DISCUSSION**

In the present study HAM method is applied to obtain an explicit analytic solution of nanofluid over a flat plate (Fig. 1). First, a comparison between the applied methods, obtained by the numerical method and HAM for nanoparticles volume fraction $\Phi = 0.3$ of $\text{Cu}$ is shown in Figures 5.

![Comparison between numerical and HAM solution](image)

**Fig. 5: The comparison between the numerical and HAM solution for $f'(\eta)$ when $\phi = 0.3$**

The numerical solution is performed using the algebra package Maple 16.0, to solve the present case. The package uses a fourth order Runge–Kutta procedure for solving nonlinear boundary value (B-V) problem [26]. Validity of HAM is shown in Table 2 and Table 3. In these tables, the %Error is defined as:

$$\% \text{Error} = \left| f(\eta)_{\text{num}} - f(\eta)_{\text{HOM}} \right|$$

(25)

The results are proved to be precise and accurate in solving a wide range of mathematical and engineering problems especially Fluid mechanic cases. This accuracy gives high confidence to us about validity of this problem and reveals an excellent agreement of engineering accuracy. This investigation is completed by depicting the effects of some important parameters to evaluate how these parameters influence on this fluid.

From a physical point of view, Figures 9 to 11 are prepared in order to see the effects of the volume fraction of nano-particle $\Phi$ and particle on the velocity profile. Figure 6 is the graphical representation of the velocity boundary layer for Cu and water mixture at different values of the solid volume fraction. It can be seen that, the thickness of the boundary layer are decreases with increasing the $\Phi$ due to adding the particles that itself leads to increase in dynamic viscosity and momentum diffusion of the fluid. Hence the velocity gradient at the plate increase which is shown for Cu/water in Figure 7.

Moreover, Figures 8 and 9 have been prepared for the variations of different nanoparticles and constant solid volume fraction on the distribution of velocity.

As we seen that, the velocity profiles are decreases for TiO2/water due to decreases densities and that the same trend is observed in Al2O3/water. While an opposite trend is observed in the presence of Cu/water due to
high density of Cu. In addition, Figures 8 and 9 show that for Cu/water the dynamic viscosity increases more and leads to a thinner boundary layer than other particles.

**Fig. 6:** Effects of volume fraction of nano-particle on the velocity profile inside the boundary layer for Cu

**Fig. 7:** Effects of volume fraction of nano-particle on the velocity gradient inside the boundary layer for Cu

**Fig. 8:** Effects of particle and water on the velocity gradient inside the boundary layer at $\Phi = 0.2$
4. Conclusion:

In this investigation, the analytical approach called Homotopy Analysis Method (HAM) has been successfully applied to find the most accurate analytical solution for the velocity distributions of a steady two-dimensional boundary-layer flow of nanofluids on a flat plate. Furthermore, the obtained solutions by proposed methods have been compared with the direct numerical solutions generated by the symbolic algebra package Maple 16. Effects of different physical parameters, such as, $\Phi$, the volume fraction of nano-particle on the velocity profiles of the problem have been investigated. The following main points can be concluded from the present study:

- The comparison shows that the HAM solutions is highly accurate and provide the rapid achievement to compute the flow velocities. Also according to the previous publications this methods is a powerful technique for finding analytical solutions in science and engineering problems.
- The effects of different solid volume fraction with three different types of nanoparticles and water as a base fluid, on the velocity are discussed. The results show that the due to the high density of Cu, adding this nanoparticles to water generates more thinner boundary layer in contrast to other nanoparticles in a process.

REFERENCES


