Dynamic Inverse DEA in the Presence of Fuzzy Data

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ABSTRACT

Dynamic data envelopment analysis (DEA) is a technique for measuring the efficiency of a set of decision making units in the presence of the time factor and inter-temporal dependence. In this paper, a method for treating the fuzzy data in dynamic DEA framework is provided. In addition, this paper studies the inverse dynamic data envelopment analysis in the presence of fuzzy data. The problem input-estimation in the dynamic DEA is studied under increasing outputs and preserving the efficiency score in the presence of fuzzy data. In this paper, we extend inverse DEA to estimate the input levels under inter-temporal dependence assumption with fuzzy data. Estimation of input levels are transformed into a fuzzy multiple-objective linear programming (MOLP) to solve, where every pare to solution of the MOLP can be considered as a forecasting tool for the desired input levels.

INTRODUCTION

Data envelopment analysis (DEA) was at first suggested by Charnes et al. [1] to estimate relative efficiencies of a set of decision-making units (DMUs) in a multiple-input multiple-output production technology. This technique has allocated to itself a wide range of research in operations research field. In DEA, it is assumed that all inputs and outputs data are precisely known.

The idea of the inverse DEA at first considered by Zhang and Cui [33] where the input increases of a DMU are measured for its given output increases under preserving the CCR efficiency. Afterwards, some scholars investigated the problem more and established various results in different frameworks, see, e.g. [6,7,12,16-20,27,28,31,32]. Wei et al. [31], Hadi-Vencheh et al. [10,11], and Jahanshahloo et al. [17] studied the problems output-estimation and input-estimation. In other words, they considered inverse DEA to answer this question: “if among a group of DMUs, the decision maker increases certain outputs to a particular unit and assumes that the DMU, with respect to other DMUs, maintains or improves its current efficiency level, how much should the inputs of the DMU increase? In addition, Jahanshahloo et al. [19] studied the problem input and output-estimation simultaneously provided that the DMU maintains its current efficiency level. They used pareto solutions of MOLP to answer to this problems.

Dynamic DEA is a technique for computing relative efficiencies of a set of decision-making units (DMUs) in the presence of the time factor and inter-temporal dependence. This inter-temporal dependence has different forms. A form is the situation at which the capital stock influences outputs over various production periods [3,20]. The another form refers to the production processes in which some of the output levels produced in a time period are used as inputs in the next period [26,29,30]. Dynamic DEA has been studied and used in a wide variety of contexts, see e.g. [4,21-24,26,29,30]. Emrouznejad and Thanassoulis [3] studied the especial causes of dynamic DEA. They suggested an LP model to estimate the technical efficiency to estimate relative efficiencies of a set of decision-making units (DMUs) in a dynamic framework. This model was revised by Jahanshahloo et al. [20]. The problems input-estimation and output-estimation are extended to a dynamic framework by Jahanshahloo et al. [18]. They applied MOLP models and necessary and sufficient conditions for input/output estimation are introduced.

Furthermore, fuzzy decision making is a suitable tool in decision-making theory. Therefore, a wide range of research in DEA field has been allocated to fuzzy DEA, see, e.g., [2,5,12-15]. Fuzzy DEA is a fuzzy...
mathematical programming-based technique for evaluating relative efficiencies of a set of decision-making units (DMUs) under imprecise data both practically and theoretically. Kao and Liu [14,15], based on the notion of fuzziness, transformed a fuzzy DEA model to a family of ordinary DEA models using the $\alpha$--cut approach. Some scholars proposed fuzzy DEA models, using concept of comparison of fuzzy numbers, see, e.g., [8,9,25]. The problem input-estimation in traditional DEA extended to fuzzy DEA by Ghobadi and Jahangiri [7] and provided a sufficient condition for efficiency maintaining in the presence of fuzzy data.

This paper studies the inverse version of dynamic DEA model addressed in Emrouznejad and Thanassoulis [3] and Jahanshahloo et al. [20] in the presence of fuzzy data. The problem input-estimation investigated under inter-temporally dependence assumption with fuzzy data. Necessary and sufficient conditions to estimate input levels are introduced based on pare to solutions of fuzzy MOLP problems.

The given results are important both theoretically and practically because it provides some of mathematical foundations between fuzzy dynamic DEA and fuzzy MOLP problems. In addition, it can help the decision maker to make better decisions in order to develop DMUs.

The reminder of this paper is organized as follows: In section 1, we review some preliminaries from fuzzy decision-making theory. In section 2, we extend dynamic DEA model, provided by Emrouznejad and Thanassoulis [3] and Jahanshahloo et al. [20], in the presence of fuzzy data. Section 3 is devoted to the main results of the paper. In this section, the input-estimation problem is dealt with in the presence of fuzzy data. Section 4 gives a brief conclusion.

1. L-R fuzzy numbers in Fuzzy Decision Making:

In this section, we review some of the basic concept of the fuzzy numbers needed through the paper. Let $X$ be a nonempty topological space. The set of all fuzzy subsets $\tilde{A}$ of $X$, we denote by $F(X)$, where every fuzzy subsets $\tilde{A}$ of $X$ is uniquely determined by the membership function $\mu_{\tilde{A}}: x \to [0,1]$. The $\alpha$--cut set of $\tilde{A}$ is the set $[\tilde{A}]_{\alpha} = \{ x \in X | \mu_{\tilde{A}}(x) \geq \alpha \}$ for each $\alpha \in (0,1]$, while $[\tilde{A}]_{0} = \text{Closure} \{ x \in X | \mu_{\tilde{A}}(x) > 0 \}$. $[\tilde{A}]_{0}$ is called support $\tilde{A}$ of and denoted by $\text{Supp}\tilde{A}$. $[\tilde{A}]_{\alpha}$ and $[\tilde{A}]_{\beta}$ are represented the lower and upper endpoint of any $\alpha$--cut set of $\tilde{A}$, respectively.

Let $L,R : [0,1] \to [0,1]$, with $L(0)=R(0)=1$ and $L(1)=R(1)=0$, are non-increasing, continuous shape functions. An L-R fuzzy number is denoted as $(\alpha, \beta, \sigma, \gamma)_{L-R}$ and defined with the following membership function:

$$
\mu_{\tilde{A}}(x) =
\begin{cases}
\frac{\beta - x}{\gamma} & \text{if } \alpha - \beta \leq x \leq \alpha,
1 & \text{if } \alpha \leq x \leq \sigma,
1 - \frac{\gamma - \beta}{\alpha - \gamma} & \text{if } \sigma \leq x \leq \sigma + \gamma,
0 & \text{otherwise},
\end{cases}
$$

where $\beta, \gamma > 0$ are positive scalers and $[\alpha, \sigma]$ is the peak of $\tilde{A}$. The set all L-R fuzzy numbers denote by $\text{FNLR}([\alpha, \beta])$. If $[\tilde{A}]_{\alpha} > 0$, then L-R fuzzy number $\tilde{A}$ is called a positive L-R fuzzy number. For each $\alpha \in (0,1]$, it is easy see that $\tilde{A}_{\alpha} = [a - L^{-1}(\alpha) \beta, a + R^{-1}(\alpha) \gamma]$. (1)

Let $\tilde{A} = (\alpha, \beta, \sigma, \gamma)_{L-R}$ and $\tilde{B} = (b, \delta, \eta, \mu)_{L-R}$ be two L-R fuzzy numbers and $\lambda$ a non-negative real number. Then $\lambda \tilde{A} + \tilde{B} = (\alpha + \lambda \eta, \beta + \lambda \mu)_{L-R}$, and $\lambda \tilde{A} = (\lambda \alpha, \lambda \beta, \lambda \sigma, \lambda \gamma)_{L-R}$.

(2)

In particular, for a given set of L-R fuzzy numbers $\tilde{A}_{j} = (\alpha_{j}, \beta_{j}, \sigma_{j}, \gamma_{j})_{L-R}, j = 1, \ldots, n$ with common shape functions (L and R) and $\lambda_{j}, j = 1, \ldots, n$ are positive scalars, we have that

$$
\sum_{j} \lambda_{j} \tilde{A}_{j} = (\sum_{j} \lambda_{j}, \sum_{j} \lambda_{j} \beta_{j}, \sum_{j} \lambda_{j} \sigma_{j}, \sum_{j} \lambda_{j} \gamma_{j})_{L-R}.
$$

(3)

where $\sum \lambda_{j} \tilde{A}_{j}$ denote the combination $\lambda_{1} \tilde{A}_{1} \oplus \lambda_{2} \tilde{A}_{2} \oplus \ldots \oplus \lambda_{n} \tilde{A}_{n}$. Therefore, for each $\alpha \in (0,1]$, we get

$$
[\sum_{j} \lambda_{j} \tilde{A}_{j}]_{\alpha} = [\sum_{j} \lambda_{j} (a_{j} - L^{-1}(\alpha) \beta_{j}), \sum_{j} \lambda_{j} (a_{j} + R^{-1}(\alpha) \gamma_{j})].
$$

(4)
Soleimani-damaneh [25] defined the weighted signed distance of \( \tilde{A} \) and \( \tilde{B} \) as follows:

\[
d(\tilde{A}, \tilde{B}) = \int_{0}^{1} [l(\tilde{A})_\alpha - [\tilde{B}]_\alpha] + [\tilde{B}]_\alpha - [\tilde{B}]_\alpha^T] d\alpha.
\]  

(5)

A decision maker can rank a pair of fuzzy numbers, and \( \tilde{A} \) and \( \tilde{B} \) using \( d(\tilde{A}, \tilde{B}) \) based on the following rules:

a) \( \tilde{A} \prec \tilde{B} \) if \( d(\tilde{A}, \tilde{B}) < 0 \),

b) \( \tilde{A} \succ \tilde{B} \) if \( d(\tilde{A}, \tilde{B}) > 0 \),

c) \( \tilde{A} \equiv \tilde{B} \) if \( d(\tilde{A}, \tilde{B}) = 0 \).

If \( \tilde{A} = (a, \overline{a}, \alpha, \beta)_L \) and \( \tilde{B} = (b, \overline{b}, \eta, \mu)_L \) are L-R fuzzy numbers with the same shape functions, then we obtained:

\[
d(\tilde{A}, \tilde{B}) = (a + \overline{a} - b - \overline{b}) + (\eta - \beta) \int_{0}^{1} L^{-1}(\alpha) d\alpha + (\gamma - \mu) \int_{0}^{1} R^{-1}(\alpha) d\alpha.
\]  

(7)

Let \( \tilde{A} = (a, \overline{a}, \alpha, \beta)_{L-R} \) be a L-R fuzzy number. If \( L(x) = R(x) = 1 - x \) and \( a = \overline{a} = \alpha = \beta = \gamma \), then \( \tilde{A} \) is called a triangular fuzzy number and denoted as \( \tilde{A} = (a, \alpha, \beta, \gamma)_{L-R} \). In addition, if \( L(x) = R(x) = 1 - x \), \( a = \overline{a} \), and \( \beta = \gamma \), then \( \tilde{A} \) is called a symmetric triangular fuzzy number.

2. Inter-temporal dependence with fuzzy data:

In this section, we extend LP problem, provided by Emrouznejad and Thanassoulis [3], to estimate the relative efficiency of an assessment path under fuzzy data. Let us assume that we have a set of \( n \) DMUs, \( \{DMU \_j : j = 1, \ldots, n\} \), whose performance is assessed in a time horizon, say, \( t = 1, 2, \ldots, n + T \). In addition considering a window of periods: \( w = \{t | t = \tau + 1, 2, \ldots, n + T\} \) as assessment window. The set of inputs, \( I = \{1, 2, \ldots, m\} \) is divided into two subsets \( I^1 \) (period-specific fuzzy inputs) and \( I^2 \) (capital fuzzy inputs), where \( I^1 \cap I^2 = \emptyset \) and \( I^1 \cup I^2 = I \). For \( j = 1, \ldots, n \), the set of inputs is considered as: period-specific fuzzy input paths \( (\tilde{\chi}^t_j, t + 1, \ldots, t + T) \), change in fuzzy stock paths \( (\tilde{\epsilon}^t_j, t + 1, \ldots, t + T) \), and initial-stock fuzzy inputs \( (\tilde{\xi}^t_j, \tilde{\eta}^t_j) \). For \( j = 1, \ldots, n \), the set of outputs, \( O = \{1, 2, \ldots, s\} \) is divided to two kinds of outputs as: fuzzy output paths \( (\tilde{y}^t_j, t + 1, \ldots, t + T) \), terminal-stock fuzzy inputs as outputs \( (\tilde{\zeta}^t_j) \). Also, it is presumed that all input-output levels are positive fuzzy number. Note that \( \tilde{\xi}^t_j \), \( \tilde{\eta}^t_j \), and \( \tilde{\zeta}^t_j \) values satisfy the following equation:

\[
\tilde{\zeta}^t_j = \tilde{\xi}^t_j + \tilde{\eta}^t_j, \quad j = 1, \ldots, n, \forall i \in I^1, (8)
\]

We consider the input-output levels in the each of \( t \in w \) as L-R fuzzy numbers as

\[
\tilde{\chi}^t_j = (\tilde{\chi}^t_j, \chi^t_j, \tilde{\chi}^t_j, \chi^t_j), \quad j = 1, \ldots, n, i \in I^1, t \in w,
\]

\[
\tilde{\zeta}^t_j = (\tilde{\zeta}^t_j, \zeta^t_j, \tilde{\zeta}^t_j, \zeta^t_j), \quad j = 1, \ldots, n, i \in I^1, t \in w,
\]

\[
\tilde{\eta}^t_j = (\tilde{\eta}^t_j, \eta^t_j, \tilde{\eta}^t_j, \eta^t_j), \quad j = 1, \ldots, n, r \in O, t \in w,
\]

\[
\tilde{\zeta}^T_j = (\tilde{\zeta}^T_j, \zeta^T_j, \tilde{\zeta}^T_j, \zeta^T_j), \quad j = 1, \ldots, n, i \in I^2,
\]

\[
\tilde{\zeta}^T_j = (\tilde{\zeta}^T_j, \zeta^T_j, \tilde{\zeta}^T_j, \zeta^T_j), \quad j = 1, \ldots, n, i \in I^2.
\]

Suppose that \( (\tilde{\chi}^t_j, \tilde{\chi}^t_j, t + 1, \ldots, t + T, \tilde{\zeta}^t_j, \tilde{\eta}^t_j) \) denote the assessment path of \( DMU \_j \); \( j = 1, \ldots, n \) in the assessment window, \( w \). Emrouznejad and Thanassoulis [3] provided an LP problem for measuring the relative efficiency of an assessment path with data exactly known. Jahanshahloo et al. [20] modified Emrouznejad and Thanassoulis' model. This model can be naturally extended to be the following fuzzy DEA model:

\[
1(\tilde{x}^t_j, \ldots, \tilde{x}^t_j) \text{ is denoted as } \tilde{x}^t_j \text{ for simplicity. Similar notations will be applied for other inputs/outputs.}
\[ \rho^*_o = \min \frac{\sum \theta_i}{T + 1} \]

\[ s.t. \quad \sum \lambda_j \hat{x}^o_j \leq \theta \bar{x}^o_i, \quad i \in I_1, \quad t \in w, \quad (9) \]

\[ \sum \lambda_j \zeta^o_j \leq \theta \bar{z}^o_i, \quad i \in I_2, \quad t \in w, \quad (10) \]

where \( \Omega = [\lambda | \lambda = (\lambda_1, \ldots, \lambda_n), \delta \left( \sum \lambda_j + \delta_j (-1)^j v \right) = \delta_j \nu \geq 0, \lambda_j \geq 0, j = 1, \ldots, n] \).

\[ \delta_j \] are parameters with \( 0 \leq 1 \) values. If \( \delta_j = 0 \), then model (9) is under a constant returns to scale (CRS), if \( \delta_1 = 1 \) and \( \delta_j = 0 \), then the above model is under a variable returns to scale (VRS), if \( \delta_1 = 1 \), then the above model is under a non-increasing returns to scale (NIRS), and if \( \delta_1 = \delta_2 = \delta_3 = 1 \), then the above model is under a non-decreasing returns to scale (NDRS) assumption of the production technology.

Using relations (3)-(6), the above model can be converted to the following optimization problem:

\[ \rho^*_o = \min \frac{\sum \theta_i}{T + 1} \]

\[ s.t. \quad d(\theta \bar{x}^o_i, \sum \lambda_j \hat{x}^o_j) \geq 0, \quad i \in I_1, \quad t \in w, \quad (9) \]

\[ d(\theta \bar{z}^o_i, \sum \lambda_j \zeta^o_j) \leq 0, \quad i \in I_2, \quad t \in w, \quad (10) \]

\[ \theta \leq 1, \lambda \in \Omega, \quad t \in w. \]

3. Inverse DEA under inter-temporal dependence with fuzzy data:

In this section, a method for treating the fuzzy data in inverse dynamic DEA framework is provided. Here, we devoted to extending Question, provided by Hadi-Vencheh et al. [11] and was extended by Jahanshahloo et al. [18], in the presence of fuzzy data. Suppose that the outputs of DMU \( o \) are increased from \( \hat{y}^{o, t \in \tau} \) to \( \hat{y}^{o, t \in \tau} + \Delta \hat{y}^{o, t \in \tau} \), where \( \Delta \hat{y}^{o, t \in \tau} \in (\text{FNLR} (\mathbb{R}^+_o))^s \). The aim of the problem is estimating the input vectors \( \hat{\phi}^{o, t \in \tau} = (\hat{\phi}^{o, t \in \tau}, \hat{\phi}^{o, t \in \tau}, \hat{\phi}^{o, t \in \tau}, \hat{\phi}^{o, t \in \tau})_{\text{N-L}, \text{R}}, \) and \( \hat{y}^{o, t \in \tau} = (\hat{y}^{o, t \in \tau}, \hat{y}^{o, t \in \tau}, \hat{y}^{o, t \in \tau}, \hat{y}^{o, t \in \tau})_{\text{N-L}, \text{R}}, \) provided that the efficiency index of DMU \( o \), with respect to other DMUs, is still \( \rho^*_o \). In fact, \((\hat{\phi}^{o, t \in \tau}, \hat{y}^{o, t \in \tau}) = (\hat{\phi}^{o, t \in \tau}, \hat{y}^{o, t \in \tau}) \), where \((\Delta \hat{y}^{o, t \in \tau}, \Delta \hat{y}^{o, t \in \tau}) \in (\text{FNLR} (\mathbb{R}^+_o))^s \).
To solve Question, based on Jahanshahloo et al. [18], we supply the increases of $Z_{io}$ values by increasing $Z_{io}^{-1}$, because $Z_{io}^{-T}$ is considered as output. Therefore the following equations are considered:

$$
\tilde{\gamma}_{o}^{-T} \approx \tilde{Z}_{o}^{-T} \oplus \Delta \tilde{Z}_{o}^{-1}, \quad \Delta \tilde{Z}_{o}^{-1} \in (FNLR \mathbb{R})_{\tilde{\gamma}^o} \tag{11}
$$

$$
\tilde{\gamma}_{o}^{-T} \approx \tilde{Z}_{o}^{-T} \oplus \sum_{i} \tilde{h}_{o}^{i} \tag{12}
$$

Suppose $DMU_{n+1}$ represents $DMU_{o}$ after changing the input-output levels. The following fuzzy model measures the efficiency score of $DMU_{n+1}$:

$$
\rho_{o}^{*} = \min \frac{\sum_{i} \theta^{i}}{T + 1}
$$

$s.t.$

$$
\sum_{j} \lambda_{j} \varepsilon_{j}^{o} \oplus \lambda_{o} \tilde{v}^{o} \leq \theta^{o} \tilde{d}_{o}^{*}, \quad i \in I_{1}, \forall t \in w, \tag{13}
$$

$$
\sum_{j} \lambda_{j} \varepsilon_{j}^{o} \oplus \lambda_{o} \tilde{v}^{o} \leq \theta^{o} \tilde{d}_{o}^{*}, \quad i \in I_{2}, \forall t \in w, \tag{13}
$$

$$
\sum_{j} \lambda_{j} \varepsilon_{j}^{o} \oplus \lambda_{o} \tilde{v}^{o} \leq \theta^{o} \tilde{h}_{o}^{i}, \quad r \in O, \forall t \in w, \tag{13}
$$

$$
\sum_{j} \lambda_{j} \tilde{Z}_{j}^{o-t} \oplus \lambda_{o} \tilde{Z}_{o}^{o-t} \geq \tilde{Z}_{o}^{o-t}, \quad i \in I_{1}, \tag{13}
$$

$$
\sum_{j} \lambda_{j} \tilde{Z}_{j}^{o-t} \oplus \lambda_{o} \tilde{Z}_{o}^{o-t} \geq \tilde{Z}_{o}^{o-t}, \quad i \in I_{2}, \tag{13}
$$

$$
\theta^{i} \leq \lambda \in \Omega, \quad \forall t \in w, \tag{13}
$$

where

$$
\Omega^{o} = \{ \lambda \mid \lambda = (\lambda_{i}, \ldots, \lambda_{n+1}), \delta_{1} \sum_{j} \lambda_{j} + \delta_{2}(-1)^{v}v = \delta_{1}v \geq 0, \lambda_{j} \geq 0, j = 1, \ldots, n+1 \}. \tag{13}
$$

If the optimal values of problems (9) and (13) are equal, it is said that the efficiency remains unchanged, i.e.,

$$
\text{eff} (\tilde{\alpha}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}) = \text{eff} (\tilde{\alpha}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}) \tag{13}
$$

To solve Question, the following fuzzy MOLP problem is considered:

$$
\min \ (\tilde{h}^{o}_{o}, i \in I_{1} \land \tilde{h}^{o}_{o}, i \in I_{1}) \forall t \in w \tag{14}
$$

$s.t.$

$$
\sum_{j} \lambda_{j} \varepsilon_{j}^{o} \leq \theta^{o} \tilde{d}_{o}^{*}, \quad i \in I_{1}, \forall t \in w, \tag{14}
$$

$$
\sum_{j} \lambda_{j} \varepsilon_{j}^{o} \leq \theta^{o} \tilde{d}_{o}^{*}, \quad i \in I_{2}, \forall t \in w, \tag{14}
$$

$$
\sum_{j} \lambda_{j} \tilde{Z}_{j}^{o-t} \geq \tilde{Z}_{o}^{o-t}, \quad r \in O, \forall t \in w, \tag{14}
$$

$$
\sum_{j} \lambda_{j} \tilde{Z}_{j}^{o-t} \geq \tilde{Z}_{o}^{o-t}, \quad i \in I_{1}, \tag{14}
$$

$$
\sum_{j} \lambda_{j} \tilde{Z}_{j}^{o-t} \geq \tilde{Z}_{o}^{o-t}, \quad i \in I_{2}, \tag{14}
$$

$$
\tilde{\gamma}_{o}^{*} \approx \tilde{Z}_{o}^{-T} \oplus \sum_{i} \tilde{h}_{o}^{i}, \quad i \in I_{2}, \tag{14}
$$

$$
\tilde{\gamma}_{o}^{*} \approx \tilde{Z}_{o}^{-T} \oplus \sum_{i} \tilde{h}_{o}^{i}, \lambda \in \Omega, \quad \forall t \in w, \tag{14}
$$

where $(\theta^{o}, t \in w)$ is an optimal solution to problem (9) and $(\lambda, \tilde{\alpha}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*})$ is the variable vector. Note that in this model, $\tilde{\gamma}_{o}^{*} \geq \tilde{Z}_{o}^{-T}$ automatically holds. The following theorem shows how the above MOLP can be used to estimate input levels.

**Theorem 3.1** Suppose that the efficiency score of $DMU_{o}$ is equal one. Let $\Lambda = (\tilde{\alpha}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*}, \tilde{\gamma}_{o}^{*})$ be a pareto solution to fuzzy problem (14) in which one of the following assumptions holds:
(i) \( \hat{\alpha}_w, \hat{\eta}_w, \hat{\Gamma}_w \approx (\hat{x}_w, \hat{z}_w, \hat{\bar{Y}}_w) \), \\
(ii) \( \hat{\beta}_w \neq \hat{\xi}_w \) for each \( w \in W \).

Then 

\[
\text{eff} (\tilde{\alpha}_w, \tilde{\eta}_w, \tilde{\Gamma}_w) = \text{eff} (\tilde{x}_w, \tilde{z}_w, \tilde{Y}_w).
\]

**Proof.** It is obvious that \( \theta^* = 1 \) for each \( t \in W \) because the efficiency score of \( DMU_t \) is equal one. Because \( \Lambda \) is a feasible solution of fuzzy MOLP (14), the following relations are held:

\[
\sum_{j=1}^{n} \lambda_j^* x_{ij}^* \leq \theta^* \hat{\alpha}^+_w, i \in I_1, \forall \ t \in w, \quad (15)
\]

\[
\sum_{j=1}^{n} \lambda_j^* z_{ij}^* \leq \theta^* \hat{\eta}_w, \quad i \in I_2, \forall \ t \in w, \quad (16)
\]

\[
\sum_{j=1}^{n} \lambda_j^* y_{ij}^* \geq \hat{\beta}_w, \quad r \in O, \forall \ t \in w, \quad (17)
\]

\[
\sum_{j=1}^{n} \lambda_j^* \tilde{z}_w \geq \tilde{\zeta}_w, \quad i \in I_2, \quad (18)
\]

\[
\sum_{j=1}^{n} \lambda_j^* \tilde{\zeta}_w \leq \tilde{\xi}_w, \quad i \in I_2, \quad (19)
\]

\[
\hat{\alpha}_w \geq \tilde{x}_w, \quad i \in I_1, \forall \ t \in w, \quad (20)
\]

\[
\hat{\eta}_w \geq \tilde{z}_w, \quad i \in I_1, \forall \ t \in w, \quad (21)
\]

\[
\hat{\lambda} \in \Omega, \quad \forall \ t \in w. \quad (22)
\]

By Eqs.(15-19) and (22), \( (\lambda^*, \theta^*; t \in W) \) is obviously a feasible solution to problem (13)(considering \( \lambda^*_w \equiv \hat{\alpha}_w, \hat{\eta}_w \equiv \hat{\eta}_w, \) and \( \theta^*_w \equiv \tilde{\zeta}_w \)), in which \( \lambda^* = (\lambda^*_0) \in \Omega \) and \( \theta^*_w, t \in W \in R^{Y* \times 1} \).

Therefore, \( \rho^*_w \leq \rho^* \). Let \( (\lambda^*, \theta^*; t \in W) \) be an optimal solution of problem (13), and the inequalities (15-19) will be used in model (13), the following results are revealed:

\[
\theta^* \hat{\alpha}_w^* \geq \sum_{j=1}^{n} \lambda_j^* x_{ij}^* \otimes \lambda_n^* \hat{\alpha}_w^* \geq \sum_{j=1}^{n} \lambda_j^* \hat{x}_{ij}^* \otimes \lambda_n^* \hat{\alpha}_w^* \geq \sum_{j=1}^{n} (\lambda_j^* \otimes \lambda_n^*) \hat{x}_{ij}^*, \quad i \in I_1, \forall \ t \in w, \quad (23)
\]

\[
\theta^* \hat{\eta}_w^* \geq \sum_{j=1}^{n} \lambda_j^* z_{ij}^* \otimes \lambda_n^* \hat{\eta}_w^* \geq \sum_{j=1}^{n} \lambda_j^* \hat{z}_{ij}^* \otimes \lambda_n^* \hat{\eta}_w^* \geq \sum_{j=1}^{n} (\lambda_j^* \otimes \lambda_n^*) \hat{z}_{ij}^*, \quad i \in I_2, \forall \ t \in w, \quad (24)
\]

\[
\hat{\beta}_w = \sum_{j=1}^{n} \lambda_j^* \gamma_{ij}^* \otimes \lambda_n^* \hat{\beta}_w = \sum_{j=1}^{n} \lambda_j^* \hat{\gamma}_{ij}^* \otimes \lambda_n^* \sum_{j=1}^{n} \lambda_j^* \hat{\gamma}_{ij}^*, \quad \hat{\beta}_w \leq \sum_{j=1}^{n} (\lambda_j^* \otimes \lambda_n^*) \hat{\gamma}_{ij}^*, \quad r \in O, \forall \ t \in w, \quad (25)
\]

\[
\tilde{Z}_w \geq \sum_{j=1}^{n} \lambda_j^* \tilde{Z}_{ij}^* \otimes \lambda_n^* \tilde{Z}_w \geq \sum_{j=1}^{n} \lambda_j^* \tilde{Z}_{ij}^* \otimes \lambda_n^* (\sum_{j=1}^{n} \lambda_j^* \tilde{Z}_{ij}^*), \quad (26)
\]

\[
\sum_{j=1}^{n} (\lambda_j^* \otimes \lambda_n^*) \tilde{Z}_{ij}^* \geq \tilde{Z}_{ij}^* \otimes 1, \quad i \in I_2, \quad (27)
\]

For each \( j = 1, \ldots, n \), set \( \tilde{\lambda}_j = \lambda_j^* + \lambda_n^* \lambda_j^* \). It is easily seen that \( \tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in \Omega \). By contradiction assume that \( \rho^*_w < \rho^* \).

If assumption (i) holds, then by Eqs.(23-27) the following inequalities are generated:
\[
\sum_{j \in T} \lambda_j \xi_j^i \leq \theta^{\nu^*} \tilde{\eta}_i \leq \theta^{\nu^*} \xi_i, \quad i \in I_1, t \in w \tag{29}
\]
\[
\sum_{j \in T} \lambda_j \gamma_j^r \geq \tilde{\rho}_r \geq \gamma_r, \quad r \in O, t \in w, \tag{30}
\]
\[
\sum_{j \in T} \lambda_j \tilde{z}^{\nu^*} \geq \tilde{z}_i, \quad i \in I_2, \tag{31}
\]
\[
\sum_{j \in T} \lambda_j \tilde{z}_j \leq \tilde{z}_i, \quad i \in I_2. \tag{32}
\]

Because \( \tilde{\lambda} \in \Omega \) and Eqs. (28-32) imply \((\tilde{\lambda}, \theta^{\nu^*} \tilde{x}^{1, \ldots, t})\) is a feasible solution to model (9). Therefore, 
\[
\rho^* \leq \frac{1}{1 + \frac{1}{\rho}} \sum_{i \in w} \theta^{\nu^*} = \rho^* < 1,
\]
which contradicts the efficiency of \( DMU \) and completes the proof in the part(i).

To prove the theorem under assumption (ii), we should show that \( \rho^* \theta^{\nu^*} = 1 \). By contradiction assumption, since \( \rho^* \theta^{\nu^*} < 1 \), then \( \frac{1}{\rho^* \theta^{\nu^*}} = 1 \), so that, there exists at least one \( p \in w \) such that \( \theta^{\nu^*} < 1 \).

Since for each \( t \in w \) and \( i \in I_1 \), \( \tilde{\alpha}_{i\alpha}^* = (\tilde{\alpha}_{i\alpha}, \tilde{\alpha}_{i\alpha}^*, \tilde{\alpha}_{i\alpha}^*) \) is a positive fuzzy number, for all \( \alpha \in [0,1] \) the following inequalities are obtained:
\[
\tilde{\alpha}_{i\alpha}^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha > 0, \quad \tilde{\alpha}_{i\alpha}^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha > 0, \quad \forall t \in w, i \in I_1.
\]
Therefore,
\[
\tilde{\alpha}_{i\alpha}^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\alpha}_{i\alpha}^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha > 0, \quad \forall t \in w, i \in I_1. \tag{33}
\]

Since for each \( t \in w \) and \( i \in I_2 \), \( \tilde{\eta}_i = (\tilde{\eta}_i^*, \tilde{\eta}_i^*, \tilde{\eta}_i^*) \) is a positive fuzzy number, in a manner similar to the proof of inequality (32), for all \( \alpha \in [0,1] \) we get
\[
\tilde{\eta}_i^* + \tilde{\eta}_i^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\eta}_i^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha > 0, \quad \forall t \in w, \forall i \in I_2. \tag{34}
\]
By (23) and (24) we have
\[
\theta^{\nu^*} (\tilde{\alpha}_{i\alpha}^* + \tilde{\alpha}_{i\alpha}^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\alpha}_{i\alpha}^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha) \geq \sum_{j=1}^n \lambda_j \tilde{x}_j^i + \sum_{j=1}^n \lambda_j \tilde{x}_j^i 
\]
\[
- \left( \sum_{j=1}^n \lambda_j \tilde{x}_j^i \right) \int_0^1 L^1(\alpha) d \alpha + \left( \sum_{j=1}^n \lambda_j \gamma_j^r \right) \int_0^1 \tilde{r}^1(\alpha) d \alpha, \quad \forall t \in w, \forall i \in I_1, \tag{35}
\]
\[
\theta^{\nu^*} (\tilde{\eta}_i^* + \tilde{\eta}_i^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\eta}_i^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha) \geq \sum_{j=1}^n \lambda_j \tilde{x}_j^i + \sum_{j=1}^n \lambda_j \tilde{x}_j^i 
\]
\[
- \left( \sum_{j=1}^n \lambda_j \tilde{x}_j^i \right) \int_0^1 L^1(\alpha) d \alpha + \left( \sum_{j=1}^n \lambda_j \gamma_j^r \right) \int_0^1 \tilde{r}^1(\alpha) d \alpha, \quad \forall t \in w, \forall i \in I_2. \tag{36}
\]
By (33), (35), \( 0 < \theta^{\nu^*} < 1 \), and \( 0 < \theta^{\nu^*} \leq 1 \) for each \( t \in w \), we get
\[
\theta^{\nu^*} (\tilde{\alpha}_{i\alpha}^* + \tilde{\alpha}_{i\alpha}^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\alpha}_{i\alpha}^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha) = \tilde{\alpha}_{i\alpha}^* + \tilde{\alpha}_{i\alpha}^* 
\]
\[
- \left( \sum_{j=1}^n \lambda_j \tilde{x}_j^i \right) \int_0^1 L^1(\alpha) d \alpha + \left( \sum_{j=1}^n \lambda_j \gamma_j^r \right) \int_0^1 \tilde{r}^1(\alpha) d \alpha, \forall t \in w \setminus \{p\}, \forall i \in I_1, \tag{37}
\]
\[
\tilde{\alpha}_{i\alpha}^* + \tilde{\alpha}_{i\alpha}^* - v_{i\alpha}^* \int_0^1 L^1(\alpha) d \alpha + \tilde{\alpha}_{i\alpha}^* + v_{i\alpha}^* \int_0^1 \tilde{r}^1(\alpha) d \alpha > \sum_{j=1}^n \lambda_j \tilde{x}_j^i + \sum_{j=1}^n \lambda_j \tilde{x}_j^i 
\]
\[
- \left( \sum_{j=1}^n \lambda_j \tilde{x}_j^i \right) \int_0^1 L^1(\alpha) d \alpha + \left( \sum_{j=1}^n \lambda_j \gamma_j^r \right) \int_0^1 \tilde{r}^1(\alpha) d \alpha, \forall i \in I_1. \tag{38}
\]
\[ \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ < \hat{\alpha}_w^\circ \cdot \forall i \in I_1. \]  
(39) \[ \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ \preceq \hat{\alpha}_w^\circ \theta^* \hat{\alpha}_w^\circ \cdot \forall t \in w \setminus \{p\}, \forall i \in I_1. \]  
(40)

Also, by (34), (36), and \( 0 < \theta^* \leq 1 \) for each \( t \in w \), we have

\[ \theta^* (\hat{\alpha}_w^* + \bar{\eta}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha) = \hat{\eta}_w^* + \bar{\eta}_w^* \]

\[ - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha \geq \sum_{j=1}^{n} \bar{X}_j \bar{z}_j^\circ + \sum_{j=1}^{n} \bar{X}_j \bar{z}_j^\circ \]

\[ = - \left( \sum_{j=1}^{n} \bar{X}_j \bar{v}_j^\circ \int_0^\infty L^{-1}(\alpha)d \alpha + \sum_{j=1}^{n} \bar{X}_j \gamma_j^\circ \int_0^\infty R^{-1}(\alpha)d \alpha \right), \forall t \in w, \forall i \in I_2. \]  
(41)

In other words,

\[ \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ \preceq \hat{\alpha}_w^* \cdot \forall t \in w, \forall i \in I_1. \]  
(42)

If assumption (ii) holds, then \( \hat{\alpha}_w^* + \hat{\alpha}_w^* \sim \bar{\eta}_w^* \). Without loss of generality, we assume that \( \hat{\alpha}_w^* + \hat{\alpha}_w^* \sim \bar{\eta}_w^* \). Therefore

\[ \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha > \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha. \]  
(43)

Now, we define

\[ R_1^\circ = \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha, \forall i \in I_1, \]

\[ R_2^\circ = \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha, \forall i \in I_1, \]

\[ \rho_1^\circ = \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha, \]

\[ \rho_2^\circ = \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha. \]

\[ \zeta = \min \{ \rho_1^\circ - R_1^\circ, \rho_2^\circ - R_2^\circ \}, \]  
(44)

If \( \zeta > 0 \), because \( R_2^\circ - R_1^\circ > 0 \) and \( \rho_2^\circ - \rho_1^\circ > 0 \). Now, define \( \hat{\theta}_w^* = \hat{\theta}_w^* + \hat{\theta}_w^* \) and \( \bar{\theta}_w^* = \bar{\theta}_w^* + \bar{\theta}_w^* \)

\[ \bar{\theta}_w^* = \left\{ \begin{array}{ll} \frac{\zeta}{2}, & \text{if } i = k, t = p, \\ \bar{\theta}_w^*, \gamma_\alpha^*, \nu_\alpha^* \end{array} \right. \]  
if \( i \in L_2, t \in w \setminus \{p\} \).

Considering (44), the following inequality is obtained:

\[ \zeta \leq R_2^\circ - R_1^\circ \Rightarrow R_1^\circ \leq R_2^\circ - \zeta, \forall i \in I_1. \]

Therefore

\[ \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ + \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ - \left( \sum_{j=1}^{n} \bar{X}_j \bar{v}_j^\circ \right) \int_0^\infty L^{-1}(\alpha)d \alpha + \left( \sum_{j=1}^{n} \bar{X}_j \gamma_j^\circ \right) \int_0^\infty R^{-1}(\alpha)d \alpha \]

\[ \leq \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha - \zeta \]

\[ = (\hat{\alpha}_w^* - \frac{\zeta}{2}) + (\hat{\alpha}_w^* - \frac{\zeta}{2}) - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha \]

\[ \leq \theta^* (\hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha), \forall i \in I_1. \]

In other words,

\[ \sum_{j=1}^{n} \bar{X}_j \bar{x}_j^\circ \preceq \theta^* \hat{\alpha}_w^* \cdot \forall i \in I_1. \]  
(45)

By (44), we have \( \zeta \leq \rho_1^\circ - \rho_2^\circ \Rightarrow \rho_2^\circ \leq \rho_1^\circ - \zeta \), then

\[ \bar{\alpha}_w^* + \bar{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha \]

\[ \leq \hat{\alpha}_w^* + \hat{\alpha}_w^* - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha - \zeta \]

\[ = (\hat{\alpha}_w^* - \frac{\zeta}{2}) + (\hat{\alpha}_w^* - \frac{\zeta}{2}) - \nu_\alpha^* \int_0^\infty L^{-1}(\alpha)d \alpha + \gamma_\alpha^* \int_0^\infty R^{-1}(\alpha)d \alpha \]  
(46)

By (20) \( \zeta \preceq \hat{\alpha}_w^* \) if \( i \neq k \) or \( i = p \) and (46), it is clear that \( \bar{x}_j^\circ \preceq \bar{\alpha}_w^* \cdot \forall t \in w, \forall i \in I_2. \)  
(47)

In addition,

\[ \bar{\theta}_w^* \approx \bar{\theta}_w^* \approx \bar{x}_j^\circ, \forall t \in w, \forall i \in I_2, \]  
(48)

\[ \bar{\lambda}_w^* = \bar{\alpha}_w^* + \sum_{j=1}^{n} \bar{\alpha}_w^*, \forall i \in I_2. \]  
(49)
Since \( \tilde{X} \in \Omega \) because of (25)-(27), (40), (42), (45), and (47)-(49), \((\tilde{\lambda}, \tilde{\alpha}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}, \tilde{\Gamma}_o^{r-1})\) is a feasible solution to problem (14), in which

\[
d(\tilde{\alpha}^*_o, \tilde{\alpha}_o) = \begin{cases} 
\zeta & \text{if } t = k, l = p, \\
0 & \text{otherwise},
\end{cases}
\]

Therefore, for each \( t \in \mathcal{W} \) we have \( \tilde{\alpha}_o^* - \tilde{\alpha}_o^* \neq \tilde{\alpha}_o^* \) and \( \tilde{x}^* \neq \tilde{x}^* \). This contradicts the assumption that \( \Lambda \) is a pareto solution to problem (14) and completes the proof under assumption (ii).

**Remark 3.2:**
It is easily seen that Theorem 3.1 is valid if one replaces the objective function of fuzzy MOLP (14) with \( "\min(\tilde{\alpha}_o, \ldots, \tilde{\alpha}_o)" \).

To illustrate the using of the methodology that extended, the following numerical example is considered.

**Example 3.3:**
Consider a problem of three DMUs (A, B, and C), such that produce an output \((\tilde{y})\), with utilize a period-specific input \((\tilde{x})\), and a capital input \((\tilde{z})\). Suppose that \( L(x) = R(x) = 1 \) and all inputs and outputs are symmetric triangular fuzzy numbers. The performance of DMUs are evaluated in a window including two periods, \( w = \{1, 2\} \). The data is written in Table 1.

<table>
<thead>
<tr>
<th>DMU</th>
<th>( x^1 )</th>
<th>( y^1 )</th>
<th>( z^1 )</th>
<th>( x^2 )</th>
<th>( y^2 )</th>
<th>( z^2 )</th>
<th>( Z^0 )</th>
<th>( Z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(6,1.0)</td>
<td>(2.5,1.0)</td>
<td>(7,1.0)</td>
<td>(4.5,1.0)</td>
<td>(5.1.0)</td>
<td>(11.1,0)</td>
<td>(35,1,0)</td>
<td>(17,1,0)</td>
</tr>
<tr>
<td>B</td>
<td>(6,1.0)</td>
<td>(5,1.0)</td>
<td>(7,1.0)</td>
<td>(4.5,1.0)</td>
<td>(7,1.0)</td>
<td>(11.1,0)</td>
<td>(35,1,0)</td>
<td>(17,1,0)</td>
</tr>
<tr>
<td>C</td>
<td>(11,5,1.0)</td>
<td>(10,1.0)</td>
<td>(11,1,0)</td>
<td>(9,1,0)</td>
<td>(14,1,0)</td>
<td>(17,5,1)</td>
<td>(35,1,0)</td>
<td>(6,5,1,0)</td>
</tr>
</tbody>
</table>

The efficiency score of DMU \( A \) (under a variable returns to scale (VRS) assumption of the production technology) is equal to one. Suppose that the decision maker is interested increasing the output vector from \((\tilde{y}, \tilde{\alpha}^*) = ((2.5,1.0),(5,1.0))\) to \((\tilde{\beta}^*, \tilde{\beta}^*) = ((6,1.0),(7.5,1.0))\). The aim is to estimate the inputs under maintaining the efficiency. Considering fuzzy MOLP model (14) corresponding to DMU, a fuzzy pareto solution is generated as follows: \((\tilde{\alpha}^*, \tilde{\eta}^*, \tilde{\alpha}_o^*, \tilde{\eta}_o^*, \tilde{\Gamma}^*) = ((6.39,1.0),(7.29,1.0),(4.82,1.0),(11.46,1.0),(35,75,1.0)).\)

Therefore, according to Theorem (3.1), we have

\[
eff(\tilde{x}^1, \tilde{\eta}^1, \tilde{\alpha}_o^*, \tilde{\eta}_o^*, \tilde{\Gamma}^*) = \eff(\tilde{x}^1, \tilde{\eta}^1, \tilde{\alpha}_o^*, \tilde{\eta}_o^*, \tilde{\Gamma}^*).
\]

Here, we extend periodic weakly Pareto (PWP) solution, introduced by Jahanshahloo et al. [18], to a fuzzy dynamic framework.

**Definition 3.4:**
Let \( \Lambda = (\tilde{\lambda}, \tilde{\alpha}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}) \) be a feasible solution to fuzzy MOLP (14). \( \Lambda \) is called a fuzzy periodic weakly Pareto (FPWP) solution to MOLP (14) if there does not exist another feasible solution \((\tilde{\lambda}, \tilde{\alpha}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}) \) and some \( p \in \mathcal{W} \) such that \((\tilde{\alpha}^*_o, \tilde{\eta}^*_o) \prec (\tilde{\alpha}_o, \tilde{\eta}_o) \) and \((\tilde{x}^*, \tilde{\eta}^*_o) \leq (\tilde{x}^*, \tilde{\eta}^*_o) \) for each \( t \neq p \).

The following theorem is converse version of Theorem 3.1. Notice that, unlike Theorem 3.1, Theorem 3.5 holds for each of DMUs without assuming the efficiency score.

**Theorem 3.5**
Let \((\tilde{\alpha}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}) \) be an optimal solution to problem (9) with the optimal value of \( \tilde{\rho}_o^* \).

Suppose that \( \hat{\Lambda} = (\hat{\lambda}, \hat{\alpha}_o^{x+r+1, \ldots, T}, \hat{\eta}_o^{r, r+1, \ldots, T}, \hat{\Gamma}_o^{r-1}) \) is a feasible solution to problem (14). If

\[
eff(\hat{\alpha}_o^{x+r+1, \ldots, T}, \hat{\eta}_o^{r, r+1, \ldots, T}, \hat{\alpha}_o^{x+r+1, \ldots, T}, \hat{\eta}_o^{r, r+1, \ldots, T}, \hat{\Gamma}_o^{r-1}) \leq \eff(\tilde{x}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}, \tilde{\alpha}_o^{x+r+1, \ldots, T}, \tilde{\eta}_o^{r, r+1, \ldots, T}, \tilde{\Gamma}_o^{r-1})
\]

then \( \hat{\Lambda} \) is a fuzzy periodic weak Pareto solution to MOLP (14).

**Proof.** If \( \hat{\Lambda} \) is not a fuzzy periodic weak Pareto solution of MOLP (14), then there exists another feasible
solution \( \tilde{\Lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \) and some \( p \in W \), such that \( \tilde{w} \prec (\lambda_1, \lambda_2, ..., \lambda_n) \) and \( (\tilde{\alpha}_a, \tilde{\lambda}_a) \) for each \( t \in W \). Because \( \tilde{w} \prec \tilde{\lambda}_a \) and \( \tilde{w} \sim \tilde{\lambda}_a \), for each \( t \in W \), therefore,

\[
\tilde{\lambda}_a + \tilde{w}_a - \tilde{w}_a \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\lambda}_a \int_0^1 R^{-1}(\alpha) d\alpha > \tilde{\lambda}_a + \tilde{w}_a
\]

By inequalities (50) and (51), we get

\[
\sum_{t=1}^n \tilde{\lambda}_a + \sum_{t=1}^n \tilde{w}_a - \left( \sum_{t=1}^n \tilde{w}_a \right) \int_0^1 L^{-1}(\alpha) d\alpha + \left( \sum_{t=1}^n \tilde{\lambda}_a \right) \int_0^1 R^{-1}(\alpha) d\alpha > \sum_{t=1}^n \tilde{w}_a
\]

Thus implies that \( d(H_{\tilde{\lambda}}(\tilde{\lambda}_a, \tilde{\lambda}_b)) > 0, \forall i \in I \). In other words,

\[
\tilde{v}_{t w} > \tilde{v}_{t w}, \forall i \in I.
\]

Feasibility of \( \tilde{\Lambda} \) for fuzzy MOLP (14), implies

\[
\sum_{j=1}^n \gamma_j \gamma_j \leq \theta \tilde{\alpha}_a, \quad i \in I, t \in w \setminus \{p\}
\]

Considering inequality (61), the following inequality is obtained:

\[
\sum_{j=1}^n \tilde{\lambda}_j \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\lambda}_a \int_0^1 R^{-1}(\alpha) d\alpha \leq \tilde{w}_a
\]

By Eqs. (54) and (63), we have

\[
\sum_{j=1}^n \tilde{\lambda}_j \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\lambda}_a \int_0^1 R^{-1}(\alpha) d\alpha < \tilde{w}_a
\]

In other words,

\[
\sum_{j=1}^n \tilde{\lambda}_j \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\lambda}_a \int_0^1 R^{-1}(\alpha) d\alpha \leq \tilde{w}_a
\]

Also, because \( \tilde{\alpha}_a < \tilde{\alpha}_a \), therefore

\[
\tilde{\alpha}_a + \tilde{\alpha}_a - \tilde{\alpha}_a \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\alpha}_a \int_0^1 R^{-1}(\alpha) d\alpha > \tilde{\alpha}_a + \tilde{\alpha}_a - \tilde{\alpha}_a \int_0^1 L^{-1}(\alpha) d\alpha + \tilde{\alpha}_a \int_0^1 R^{-1}(\alpha) d\alpha
\]
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha) \quad \forall i \in I_1, 
\]
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha) \quad \forall i \in I_1, 
\]
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha) \quad \forall i \in I_2. 
\]

Considering inequalities (50) and (66)-(68), the following inequalities are obtained:
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha), \quad \forall i \in I_1, 
\]
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha) \quad \forall i \in I_2. 
\]

By inequalities (69) and (70), there exists positive scalar \( \zeta^p < 1 \) such that
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\zeta^p \theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha), \quad \forall i \in I_1, 
\]
\[
\sum_{j=1}^{n} \lambda_j x_j^p + \sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 L^1(\alpha)d\alpha + \left( \sum_{j=1}^{n} \lambda_j y_j^p \right) \int_0^1 R^{-1}(\alpha)d\alpha \leq \\
\zeta^p \theta^p(\tilde{\lambda}_w^p + \tilde{\lambda}_w^p - \tilde{v}_w^p \int_0^1 L^1(\alpha)d\alpha + \tilde{\gamma}_w^p \int_0^1 R^{-1}(\alpha)d\alpha) \quad \forall i \in I_2. 
\]

In other words,
\[
\sum_{j=1}^{n} \lambda_j x_j^p \leq (\zeta^p \theta^p) \tilde{\lambda}_w^p, \quad i \in I_1, 
\]
\[
\sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq (\zeta^p \theta^p) \tilde{\lambda}_w^p, \quad i \in I_2. 
\]

Because \((\tilde{\lambda}_w^p, \tilde{\lambda}_w^p) \leq (\tilde{\lambda}_w^p, \tilde{\lambda}_w^p) \forall \in w \in \{p\}\); Eqs. (55-56) and following a manner similar to the proof of the relations (69) and (70), the following inequalities are obtained:
\[
\sum_{j=1}^{n} \lambda_j x_j^p \leq \theta^p \tilde{\lambda}_w^p, \quad i \in I_1, t \in w \in \{p\} \quad (75) 
\]
\[
\sum_{j=1}^{n} \lambda_j \bar{x}_j^p \leq \theta^p \tilde{\lambda}_w^p, \quad i \in I_2, t \in w \in \{p\}. \quad (76) 
\]

According to Equations (59-60), (62), (65), and (73)-(76), the vector \((\tilde{\lambda}_w = \tilde{\lambda}_w = 0) \in \Omega^p\).
\[
\theta^p = \zeta^p \theta^p, \quad \theta^p = \theta^p \quad (76) 
\]

Therefore,
\[
eff (\tilde{\lambda}_w^p, \tilde{\lambda}_w^p, \tilde{v}_w^p, \tilde{\gamma}_w^p) = \rho^p \quad (76) 
\]

This contradicts the assumption and completes the proof.

4. Conclusion

In the present paper, we extended the problem input-estimation in the presence of fuzzy data to a dynamic framework. Necessary and sufficient conditions were established using pare to solutions of fuzzy MOLP under preserving the efficiency score. The suggested results are established using the weighted signed distance of fuzzy data. Although we have considered only the problem input-estimation for extending, the problem output-
estimation can be dealt with similarly after some simple modifications. Note that that many of the current models/approaches are suffering from computational and increases the computational complexity. However, the approach used in the present paper at first the fuzzy model is transferred to a mathematical programming model, while the number of constraints will not increase.

REFERENCES


