Population Growth, a Probabilistic Approach

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ABSTRACT

Controlling the population necessitates attention to all of its effective factors. This care must have harmony to the real life. Modifying the beliefs of peoples is one of the various techniques that can be affected on this phenomenon. In this study, by a mathematical model using probability theory, the effects of some cultural beliefs of the peoples about the number of boys' children, in the growth of their population are investigated. Then by using the results of the model and Verhulst equation, a limiting number of populations with respect to their rates of collaboration and competition will be indicated. Finally, through the theory of ordinary differential equations, a general formula for this limit will be presented.

INTRODUCTION

The subject of population growth is one of the most important subjects in social development of countries. Excessive of population is not favorable as good as its reduction, since it is also worrisome [1]. In one hand, some factors such as hazard of land subsidence due to dropping water levels in many plains of the ground surface [2], destruction of forests [3], and poverty [4] make the international community encourage for population decline politics, and on the other, the requirement of Human societies to the economical development and independence, confirm their needs to young and energetic population. Population growth has been associated with many factors. Fertility and mortality rates, medical advances, marriage, divorce, the age of mothers in their first childbirth, contraception, the amount of families' incomes, war, immigration, climate changes and religious educations are some of these factors which is heretofore discussed about them [5, 6, 7, 8, 9, 10]. These factors are not independent of each other. For example, new advances in medicine cause the decreasing in mortality, increasing fertility and population growth [10]. When the mortality is high, so is the birth rate. Cheap methods of contraception cause a rapid decreasing in fertility. Increasing the incomes and financial security is an effective factor in reducing fertility [11]. The other factors which can play an effective roll in the growth phenomena are some social and cultural beliefs. Such beliefs which belong to different ethnic minorities or whole peoples live in a country are among factors that can effect on the rate of growth population. Verhulst (1804-1849) the Belgian Sociologist believed that the population of any specie of life is effected under the collaboration and competition. According to his theory, the law of the growth of population is of the form \( x_{n+1} = qx_n - rx_n^2 \), in which \( x_i \) denotes the population of the \( i^{th} \) generation and \( q, r \) are the rates of collaboration and competition respectively and vary from one society to another [12]. Here, a mathematical model indicating the effort of some beliefs in decreasing, constancy or increasing the population of an ethnic minority, or whole peoples with common beliefs of a country, in the crossing stage from one generation to another, will be provided. Then assuming the constancy of two coefficients of Verhulst equation, a solution for the equation will be obtained, and then the effects of some beliefs on the behaviors of the population will be investigated.

2. The mathematical model:
   In ancient times, in aristocratic and kingly families, the annihilation of a generation has been reckoned as a tragedy. Thus a boy was a favor for them; a custodian of their personality and greatness.

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In the years when the agricultural grounds were planted without machine, every boy was reckoned as a divine favor. Because a boy could be affective in getting revenue and crop, an assistant for his father and a custodian for his heritage and jurisdiction after him [13]. Many persons were died or killed in the tribal wars which were occurred in frequently. Since a person must be associate in these wars from each family, the existence of a man which could accepted his duties in his absence was essential for supervision of his families [14]. From their view point the only remedy way having at least two boys.

Without all of the motives which we consider for the increasing the number of children in families, their view point are an affective factor in the growth of population. If the average of the number of children in the families of a society is one, then the number of population were decrease in the cross stage from one generation to another, because two persons (parents) is replaced by one (their child). By a similar proof, if the average of the number of children of families is two, then the number of population of that society does not change, but if this average is three or more, the number of population will be increased.

Here the probability of the birth of a boy is assumed to be 0.5 and the number of the children of the families is denoted by the random variable \( X \) [15]. At first some theorems on number theory will be established.

**Theorem 2.1:**

Let \( n \in \mathbb{N} \), then

\[
\sum_{k=1}^{n} k 2^{-k} = 2 - \frac{n+2}{2^{n}} \tag{1}
\]

**Proof:**

If \( n = 1 \) then (1) ishold. Let the Theorem is true for \( n = l \), then

\[
\sum_{k=1}^{l} k 2^{-k} = 2 - \frac{l+2}{2^{l}} \tag{2}
\]

If \( n = l + 1 \) we have from (2),

\[
\sum_{k=1}^{l+1} k 2^{-k} = \sum_{k=1}^{l} k 2^{-k} + (l + 1)2^{-l-1} = 2 - \frac{l+2}{2^{l}} + \frac{l+1}{2^{l+1}} = 2 - \frac{l+3}{2^{l+1}}
\]

and (3) completes the induction [16].

**Theorem 2.2:**

Let \( n \in \mathbb{N} \), then

\[
\sum_{k=1}^{n} k(k-1)2^{-k} = 4 + \frac{(n+1)(n+2)}{2^{n}} - \frac{n^{2} + n + 1}{2^{n-1}} - \frac{n+1}{2^{n-2}} \tag{4}
\]

**Proof:**

If \( n = 2 \) then (4) is hold. Let the Theorem is true for \( n = l > 2 \), then

\[
\sum_{k=1}^{l} k(k-1)2^{-k} = 4 + \frac{(l+1)(l+2)}{2^{l}} - \frac{l^{2} + l + 1}{2^{l-1}} - \frac{l+1}{2^{l-2}} \tag{5}
\]

If \( n = l + 1 \) then we have from (5),

\[
\sum_{k=1}^{l+1} k(k-1)2^{-k} = \sum_{k=1}^{l} k(k-1)2^{-k} + (l + 1)2^{-l-1}
\]

\[
= 4 + \frac{(l+2)(l+3)}{2^{l+1}} - \frac{(l+1)^{2} + (l+1) + 1}{2^{l-1}} - \frac{(l+1)+1}{2^{l-1}} \tag{6}
\]

and (6) completes the induction.

**Theorem 2.3:**

Let \( n \in \mathbb{N} \), then

\[
\sum_{k=1}^{n} k^{2}2^{-k} = \frac{9}{2} + \frac{(n+1)(n+2)}{2^{n}} - \frac{n^{2} + n + 1}{2^{n-1}} - \frac{5n+6}{2^{n}} \tag{7}
\]

**Proof:**

If \( n = 3 \) then (7) is hold. Let the Theorem is true for \( n = l > 2 \), then

\[
\sum_{k=3}^{l} k^{2}2^{-k} = \frac{9}{2} + \frac{(l+1)(l+2)}{2^{l}} - \frac{l^{2} + l + 1}{2^{l-1}} - \frac{5l+6}{2^{l}} \tag{8}
\]
If \( n = l + 1 \), then (5) implies that,

\[
\sum_{k=3}^{l+1} k^2 2^{-k} = \sum_{k=3}^{l} k^2 2^{-k} + (l + 1)^2 2^{1-l-1} = \frac{9}{2} + \frac{(l+2)(l+3)}{2^{l+1}} - \frac{(l+1)^2 + (l+1) + 1}{2^l} - \frac{5(l+1)+6}{2^{l+1}}
\]

and (9) completes the induction.

The following theorems continue the subject through the probability theory [15].

**Theorem 2.4:**

The mathematical expectation of the number of children of the families, whose final child is a boy for the first time, is equal to 2.

**Proof:**

The probability that the final boy of an \( n \)-children family is a boy is equal to \( 2^{-n} \), so the mathematical expectation of \( X \) is equal to,

\[
E(X) = \sum_{n=1}^{\infty} n 2^{-n} = \lim_{n \to \infty} \sum_{k=1}^{n} k 2^{-k} = \lim_{n \to \infty} (2 - \frac{n+2}{2^n}) = 2
\]

by Theorem 2.1.

So the population of the societies which the purpose of their families is a boy, has not a considerable growth and \( q = 1 \).

**Theorem 2.5:**

The mathematical expectation of the number of children of the families which they have one boy and one girl for the first time is equal to 3.

**Proof:**

The probability that an \( n \)-children family has both boy and girl for the first time is equal to \( 2^{1-n} \). So the mathematical expectation of \( X \) is equal to,

\[
E(X) = \sum_{n=2}^{\infty} n 2^{1-n} = 2 \sum_{n=2}^{\infty} n 2^{-n} = 2(\sum_{n=1}^{\infty} n 2^{-n} - \frac{1}{2}) = 2(2 - \frac{1}{2}) = 3
\]

by Theorem 2.1.

Consequently the population growth rate of the societies which the purpose of their families is one boy and one girl for the first time, in crossing stage from one generation to another is 50\% and \( q = 1 \cdot 5 \).

**Theorem 2.6:**

The mathematical expectation of the number of children of the families which have two boys for the first time is equal to 4.

**Proof:**

The probability that an \( n \)-children family have two boys for the first time is equal to \( (n - 1)2^{-n} \). So the mathematical expectation of the number of such families is equal to,

\[
E(X) = \sum_{n=2}^{\infty} n(n-1) 2^{-n} = \lim_{n \to \infty} (4 + \frac{(n+1)(n+2)}{2^n} - \frac{n^2+n+1}{2^{n-1}} - \frac{n+1}{2^{n-2}}) = 4
\]

by Theorem 2.2.

Thus in this case the population growth rate in crossing stage from one generation into another is 100\% and \( q = 2 \).

**Theorem 2.7:**

The mathematical expectation of the number of children of the families which have two boys and one girl for the first time is equal to 4 \( \cdot \) 5.

**Proof:**

The probability is equal to \( n 2^{-n} \) and the mathematical expectation is

\[
E(X) = \sum_{n=3}^{\infty} n^2 2^{-n} = \lim_{n \to \infty} \left( \frac{9}{2} + \frac{(n+1)(n+2)}{2^n} - \frac{n^2+n+1}{2^{n-1}} - \frac{5n+6}{2^n} \right) = \frac{9}{2}
\]
by Theorem 2.3.
Therefore in this case the population growth rate in crossing stage from one generation into another is 125% and \( q = 2 \cdot 25 \).

**Theorem 2.8:**
The mathematical expectation of the number of children of the families which have two boys and two girls for the first time is equal to 5 \( \cdot 5 \).

**Proof:**
In this case the probability is equal to \((n - 1)2^{1-n}\) and the mathematical expectation is

\[
E(X) = \sum_{n=4}^{\infty} n(n - 1)2^{1-n} = 2\lim_{n \to \infty}(\frac{11}{4} + \frac{(n+1)(n+2)}{2^n} - \frac{n^2+n+1}{2^{n-1}} - \frac{n+1}{2^n-2}) = \frac{11}{2} \tag{14}
\]

by Theorem 2.2.
Therefore in this case the population growth rate in crossing stage from one generation into another is 175% and \( q = 2 \cdot 75 \).

**RESULTS AND DISCUSSION**

The previous Theorems and the calculations (10), (11), (12), (13) and (14) show that \( q \), the rate of collaboration, can take the values \( q = 1, 1 \cdot 5, 2, 2 \cdot 25 \) and \( 2 \cdot 75 \). From now on let the competition rate assumed to be constant and opposite to zero because of different factors like disaster and war. Now note that for each value of \( q \), the number of population has an upper bound. In fact the Verhulst equation implies that

\[
4rx_{n+1} = 4rqx_n - 4r^2x_n^2 = q^2 - (2rx_n - q)^2 \leq q^2 \tag{15}
\]

therefore \( 4rx_{n+1} \leq q^2 \). So for all of values of \( q \) which for them the population grows increasingly from one generation into next, the number of population does not tend to infinity. In the following, we investigate the tide of population in each case of the above cases. Let \( r \neq 0 \) in all of these cases.

**Case 1:** \( q = 1 \):
The Verhulst equation change to \( x_{n+1} = x_n(1 - rx_n) \). Since \( 1 - rx_n < 1 \), then the sequence \( \{x_n\}_{n \in N} \) is decreasing, it is bounded below and tends to zero. In other words the population of such a society will overthrow in future.

**Case 2:** \( q = 1 \cdot 5 \):
The Verhulst equation changes to \( x_{n+1} = x_n(1 \cdot 5 - rx_n) \), so if the number of some generation is \( x_n = \frac{1}{2r} \), then all of the numbers of next generations will be the same and the number of population is constant. For the next observation let us to write the equation in the form

\[
\frac{1-2rx_{n+1}}{1-2rx_n} = 1 - rx_n \tag{16}
\]

now (15) implies that \( x_n \leq \frac{q^2}{4r} < \frac{1}{r} \) for \( n \geq 2 \), so if the number of some generation of population is less (res. more) than \( \frac{1}{2r} \), the numbers of the next generations are also less (res. more) than \( \frac{1}{2r} \) and \( \{x_n\}_{n \in N} \) is increasing (res. decreasing). In each of two previous cases the number of population tends to \( \frac{1}{2r} \).

**Case 3:** \( q = 2 \):
The Verhulst equation changes to \( rx_{n+1} - 1 = (rx_n - 1)(1 - rx_n) \), so if the number of some generation is \( x_n = \frac{1}{r} \), then all of the numbers of next generations will be the same and the number of population is constant.
Since \( x_n \leq \frac{q^2}{4r} = \frac{1}{r} \) for \( n \geq 2 \) and

\[
\frac{1-rx_{n+1}}{1-rx_n} = 1 - rx_n \tag{17}
\]
so \( \{x_n\}_{n \in N} \) is increasing and \( 1 - rx_{n+1} \) is closer to zero than \( 1 - rx_n \) and the number of population tends to \( \frac{1}{r} \) as \( n \) tends to infinity.
Case 4: \( q = 2 \cdot 25 \):

The Verhulst equation changes to \( r x_{n+1} - 1 \cdot 25 = (r x_n - 1 \cdot 25)(1 - r x_n) \), so if the number of some generation is \( x_n = \frac{5}{4r} \) or \( x_n = \frac{1}{r} \) then all of the numbers of next generations are \( \frac{5}{4r} \) and the number of population will be constant. In this case \( x_n \leq \frac{q^2}{4r} < \frac{81}{64r} \) for \( n \geq 2 \) and

\[
\frac{1.25 - r x_{n+1}}{1.25 - r x_n} = 1 - r x_n
\]  

(18)

thus \(-1 < 1 - r x_n < 1\) for \( n \geq 2 \), and (18) implies that \( 1 \cdot 25 - r x_{n+1} \) is closer to zero than \( 1 \cdot 25 - r x_n \). Moreover \( x_{n+1} > x_n \) (res. \( x_{n+1} < x_n \)) if and only if \( x_n < \frac{5}{4r} \) (res. \( x_n > \frac{5}{4r} \)) so the number of population tends to \( \frac{5}{4r} \) as \( n \) tends to infinity.

Case 5: \( q = 2 \cdot 75 \):

The Verhulst equation changes to \( r x_{n+1} - 1 \cdot 75 = (r x_n - 1 \cdot 75)(1 - r x_n) \), so if the number of some generation is \( x_n = \frac{7}{4r} \) or \( x_n = \frac{1}{r} \) then the number of population will be constant. In this case \( x_n \leq \frac{q^2}{4r} < \frac{121}{64r} \) for \( n \geq 2 \) and

\[
\frac{1.75 - r x_{n+1}}{1.75 - r x_n} = 1 - r x_n
\]  

(19)

thus \(-1 < 1 - r x_n < 1\) for \( n \geq 2 \), and (19) implies that \( 1 \cdot 75 - r x_{n+1} \) is closer to zero than \( 1 \cdot 75 - r x_n \). Moreover \( x_{n+1} > x_n \) (res. \( x_{n+1} < x_n \)) if and only if \( x_n < \frac{7}{4r} \) (res. \( x_n > \frac{7}{4r} \)) so the number of population tends to \( \frac{7}{4r} \) as \( n \) tends to infinity.

4. Differentiable stage:

Note that in all of the above cases \( x_{n+1} - x_n = (q - 1)x_n - r x_n^2 \). Therefore if \( x = x(t) \) the number of population of a society, be a differentiable function of the stage \( t \in R \), and the above equation be extended to small dimensions of time, then

\[
\frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = (q - 1)x(t) - r x(t)^2
\]  

(20)

and consequently when \( \Delta t \to 0 \), (20) changes to

\[
\dot{x}(t) = (q - 1)x(t) - r x(t)^2
\]  

(21)

A simple calculation shows that \( x(t) = \frac{q^{-1}}{A(q-1)\exp[-(q-1)t]+r} \) with \( A = \frac{1}{x(0)} - \frac{r}{q-1} \) is the only solution of (21) [17]. In the cases that \( q = 1 \cdot 5, 2, 2 \cdot 25, 2 \cdot 75 \) is greater than 1 and \( t \to \infty \), then \( x(t) \) tends to \( \frac{q^{-1}}{r} \) which is a fixed point of the Verhulst equation. Thus the population could not be tends to infinity. Moreover the limited population of a society is a decreasing (res. increasing) function of the competition (res. collaboration) rate.

5. Conclusion:

In this study, we investigated the effect of culture on population growth, allowing us to estimate the relationship between the beliefs of the ethnic minorities, or whole peoples with common beliefs, and the growth of their population. The probabilistic model produced here, is based on the cultural component, and has demonstrated that it can be applied to forecasting population growth rates. In the most today's societies, which the internal reduction factors of \( r \) and increasing factor of \( q \) still exist, the subject of population growth is very serious. The provided mathematical model shows that in addition to medical advances, health and nutrition, some ancient beliefs of peoples are effective on the increasing of the rate of collaboration and so in the growth of population. If the rate of collaboration is \( 1 \cdot 5 \) and the number of initial population is bounded above by \( \frac{1}{2r} \) then the next generations of population increases and remains bounded by the same upper bound. In this case, if the number of initial population is bounded below by \( \frac{1}{2r} \), then the next generations of population decreases and remains bounded by the same lower bound. If the rate of collaboration is 2, the population increases but does not exceed from \( \frac{1}{r} \). If the rate of collaboration is \( 2 \cdot 25 \) (res. \( 2 \cdot 75 \)), and if the number of population of a generation is less than \( \frac{5}{4r} \) (res. \( \frac{7}{4r} \)), then it will be increasing in the next one, and if the number of population of a generation...
is more than $\frac{5}{4r}$ (res. $\frac{7}{4r}$), it will be decreasing in the next stage. Therefore the variation of the number of population will be alternative. In all of these cases the number of population is bounded above by $\frac{2}{r}$.

REFERENCES